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Bethe's Lumped Constants for Small Irises

by

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Revisions to:

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Since preparation of these notes the following points have been clarified:

Page 5, last paragraph: Bethe rejects the $\overline{K}\phi$ and $(\eta/\epsilon)\nabla_f\phi$ terms of equations (9) and $\overline{K} \times \nabla_f\phi$ term of (10) in obtaining (15) and (16) on the basis that the differential contributions of these terms fail to satisfy the boundary conditions on the screen. He then, without explanation, doubles the value of $\overline{K}_m$ and η_m. The correct justification for this step is that although some differential elements in equations (9) and (10) fail to satisfy the boundary conditions, the integrated values (i.e. over the whole plane) satisfy the boundary conditions so that these terms cannot be discarded on the boundary condition argument. However in this problem the integrated contribution of $\overline{K}$, η over the whole screen (including hole) equals the contribution of $\overline{K}$, η_m over the plane (zero except in hole), hence the $\overline{K}$, η terms can be replaced by equivalent $\overline{K}_m$, η_m terms, which doubles $\overline{K}_m$, η_m in equations (15) and (16).

Page 6, last paragraph: The above revision which is a more fundamental procedure substitutes multiplying $\overline{K}_m(r')$ by two instead of dividing $\overline{F}(0)$ by two, yet giving the same final results.

Page 7, second paragraph: The reversing of the sign of one field (on left) is unecessary provided the logic of correction to page 5 is followed.

Page 7, footnote+: Bourgin[17] in the meantime pointed out that discontinuties in currents and charges around the edge of the hole were not accounted for. Bouwkamp[3] specified this condition mathematically in the form of equation (23).

Page 33A: "Indirect Measurement" means that the point for d= 0.125" was for an iris on the output of a resonator instead of the input as was done with the other sizes. It is planned to make a more reliable measurement of this when the equipment is available again.

May 15, 1951. F. B. Wood

Bethe's Lumped Constants for Small Irises

The development of the theory of diffraction for small holes and application to irises in waveguides and cavity resonators, published by H. A. Bethe in M. I. T. Radiation Laboratory Reports V-15S(128)[1A] and 43-22(194)[2], is here summarized in rationalized M. K. S. units instead of the original Gaussian unrationalized units. It is not safe to simply transform Bethe's formulas from Gaussian unrationalized units to rationalized M. K. S. units, because he defines magnetic current density as $\overline{K} = \frac{\overline{E} \times \overline{n}}{2\pi}$ yet there are some equations in 43-22 where the 2π has been dropped and one equation where a 4π required in the unrationalized system is omitted.

For the circular aperture the correct value of the H component of magnetic current given by Bouwkamp[3] is included. This error does not, however, change the radiation field and consequently it makes no change in the lumped constant polarizabilities obtained by Bethe.

As in 43-22, the equivalent magnetic and electric polarizabilities are used to obtain the Poynting vector, change of resonant frequency of a cavity, susceptance of an iris in a waveguide, and energy emitted from a cavity through aperture into free space and into a waveguide. These results differ from Bethe's results by functions of 2 and/or 2π due to discrepancies mentioned above. The iris susceptance is compared with other reliable results. The frequency shift and Q_c (coupled Q) for coupling to TE_{10} waveguide from TM_{020} cavity resonator are compared with experimental data for which the correspondence is fair. (The experimental data is from Cavity measurements which were not designed to give a direct test of Bethe's theory.)

An example of emission through an iris into free space is calculated numerically, which shows that Bethe's statement "That this power is about 25 times greater for emission into free space than for emission into a waveguide of customary dimensions [illegible] valid. No experimental data has been obtained to check [illegible].

Table of Contents

1. The Diffraction Problem.

The diffraction of electromagnetic waves by a small hole in an infinite plane conducting screen is studied first, since it is a simpler [illegible] of the problem of coupling from a resonant cavity to waveguide.

In the usual Kirchhoff method, the diffracted field is expressed in terms of the incident field in the hole, which, however, does not satisfy the boundary condition. A number of writers have set up a vector [illegible] of Kirchhoff's method,[5] but this still does not satisfy the boundary conditions when the integrals are taken over the aperture only.[6] There [illegible] rigorous solution by Sommerfeld for the diffraction of a semi-infinite [illegible], with which comparisons can be made for checking the plausibility of [illegible]. [illegible] Bouwkamp[6] has obtained a [illegible] solution for [illegible] [illegible] the [illegible] of Bethe's in the [illegible].

In the [illegible] H [illegible] [illegible] [illegible] field [illegible] e [illegible] field [illegible] h [illegible] E_0 and H_{0} [illegible] the left and the right.

[illegible] ho[illegible]'s [illegible]

[illegible] $y = u$

[illegible] Then

[illegible] of

[illegible]

$$u_p(r) = \frac{1}{4\pi} \iint \left[-\frac{\partial u_0}{\partial z'} \phi(r,r') + u_0(r') \frac{\partial \phi}{\partial z'} \right] [illegible] \quad \frac{e^{-jkr}}{r}$$

[illegible] surface

$z' = 0$, but [illegible] h. [illegible] to

assume [illegible] hole and u_0 [illegible] on the [illegible]

Then,

(5) $$u_p(r) \approx -\frac{1}{4\pi} \left[\frac{\partial u_0}{\partial z'} \phi(r) + u_0 \frac{\partial \phi(r)}{\partial z} \right]$$

for small hole and large r.

There are two alternative assumptions:

(a) $u_{0\,hole} = u_{0\,incident}$, then $u_0 \neq 0$, $\frac{\partial u_0}{\partial z'} \neq 0$ in the hole, $\frac{\partial \phi(r)}{\partial z} = 0$ for $r \gg r'$ on screen.

(b) $u_{0\,hole} = u_{0\,incident} +$ [illegible] then $u_0 = 0$, $\frac{\partial u_0}{\partial z'} = 2\frac{\partial u_0}{\partial z'}$ [illegible] hole $\frac{\partial \phi(r)}{\partial z} = 0$ for $r >$ [illegible] on screen.

So on screen at large r:

(a) $u_{p}(r) = -\frac{A}{4\pi}\frac{\partial u_0}{\partial z'}\phi(r) \neq 0$ (b) $u_{p}(r) = -\frac{A}{2\pi}\frac{\partial u_0}{\partial z'}\phi(r) \neq 0$

Since $u = E_{z}$, a component of $\overline{E}\times\overline{n}$, the boundary condition $\overline{E}\times\overline{n} = 0$ is violated.

Next we consider the vector equivalent of Kirchhoff's Theory by the direct integration of Maxwell's equation using the vector analogue of Green's Theorem as given by Stratton[9] or by [illegible][10].

Maxwell's equations and equations of continuity for $e^{j(\omega t - kr)}$

$$\text{(a)}\quad \nabla\times\overline{E} + j\omega\mu\overline{H} = -\overline{J}_{m} \qquad \text{(b)}\quad \nabla\times\overline{H} - j\omega\epsilon\overline{E} = \overline{J}$$

$$\text{(6)}\qquad \text{(c)}\quad \nabla\cdot\overline{H} = \rho_{m}/\mu \qquad \text{(d)}\quad \nabla\cdot\overline{E} = \rho/\epsilon$$

$$\text{(e)}\quad \nabla\cdot\overline{J}_{m} + j\omega\rho_{m} = 0 \qquad \text{(f)}\quad \nabla\cdot\overline{J} + j\omega\rho = 0$$

Using equations (108, 109, 111 [illegible]) we have at point [illegible]

$$\text{(a)}\quad \overline{E}(r) = \begin{cases} -\frac{1}{4\pi}\int_{V}\left(j\omega\mu\overline{J}\,\phi + \overline{J}_{m}\times\nabla_{s}\phi - \frac{\rho}{\epsilon}\nabla_{s}\phi\right)dV \\ +\frac{1}{4\pi}\int_{S}\left[-j\omega\mu(\overline{n}\times\overline{H})\phi + (\overline{n}\times\overline{E})\times\nabla_{s}\phi + (\overline{n}\cdot\overline{E})\nabla_{s}\phi\right]dS \end{cases}$$

$$\text{(b)}\quad \overline{H}(r) = \begin{cases} -\frac{1}{4\pi}\int_{V}\left(j\omega\epsilon\overline{J}_{m}\phi - \overline{J}\times\nabla_{s}\phi - \frac{\rho_{m}}{\mu}\nabla_{s}\phi\right)dV \\ +\frac{1}{4\pi}\int_{S}\left[j\omega\epsilon(\overline{n}\times\overline{E})\phi + (\overline{n}\times\overline{H})\times\nabla_{s}\phi + (\overline{n}\cdot\overline{H})\nabla_{s}\phi\right]dS \end{cases}$$

The electric and magnetic surface currents and surface charges are defined

$$\text{(8)}\quad \begin{cases} \overline{K} = \overline{n}\times\overline{H} & \eta = \epsilon(\overline{n}\cdot\overline{E}) \\ \overline{K}_{m} = \overline{E}\times\overline{n} & \eta_{m} = \mu(\overline{n}\cdot\overline{H}) \end{cases}$$

The corresponding values in [illegible] are:

$$\overline{K} = \frac{\overline{n}\times\overline{H}}{4\pi} \qquad \overline{K}_{m} = \frac{\overline{E}\times\overline{n}}{4\pi} \qquad [illegible] \quad \overline{K}_{m} = \frac{\overline{E}\times\overline{n}}{[illegible]}$$

[illegible] $\overline{F}$

Since in this problem [illegible] source are in the [illegible] of the screen, [illegible] for [illegible]

$$E(r) = \frac{1}{4\pi} \int_S [illegible] \times [illegible] \nabla \phi$$

$$H(r) = \frac{1}{4\pi} \int_S [illegible]$$

Equations [illegible] are [illegible] the complete boundary [illegible] the "vector equal[illegible]" [illegible] that [illegible] assum[illegible] [illegible]

3. Mathematical Formulation and Boundary [illegible].

Let $\overline{H}_0$ and $\overline{E}_0$ be the standing [illegible] field in the left hand side [illegible] the screen if there is no hole. The [illegible] condition at [illegible] 0 is [illegible]

(11a) $\quad \overline{n} \times \overline{E}_0 = 0$

(11b) [illegible] which make $\overline{n} \cdot \overline{H}_0$ [illegible]

H_{0tan} and $E_{0normal}$ [illegible] for [illegible] zero. [illegible] the diffracted field on the left by [illegible] on the right by $\overline{H}_2$, $\overline{E}_2$ [illegible] total field with a hole in screen [illegible]

$$\left.\begin{array}{l}\overline{H} = \overline{H}_0 + \overline{H}_1 \\ \overline{E} = \overline{E}_0 + \overline{E}_1\end{array}\right\} \; z [illegible] \qquad \begin{array}{l}\overline{H} = \overline{H}_2 \\ \overline{E} = \overline{E}_2\end{array} \quad z > [illegible] \tag{12}$$

[illegible] (11) or (12) [illegible] since there [illegible] no [illegible] in the hole:

(13) [illegible]

[illegible]

[illegible] $_{1z}(-z) = -E_{2z}(+z)$ [illegible]

[illegible]

(14a) [illegible] tan [illegible] tan [illegible] 0 [illegible]

(14b) [illegible] normal = [illegible] 0 normal [illegible] 0 [illegible]

* In [illegible]

** The [illegible]

[illegible]

The problem now is to calculate $\overline{E}_2$, $\overline{H}_2$ subject to b.c. (11a) on screen and (14a,14b) in the hole. Bethe states "These conditions are valid irrespective of size and shape of hole."[2] This statement of Bethe's has been reviewed by Silver[13] and these conditions (14) are found to correspond to the unique solution of the problem. Since $\frac{1}{2} H_{0tan} = H_{itan}$ and $\frac{1}{2} E_{on} = E_{in}$ Eq. (14) is equivalent to this statement of Smythe: "When any form of electromagnetic wave strikes a thin plane perfectly conducting sheet of any shape, the normal electric and tangential magnetic fields of the original wave are unperturbed in the apertures."

Instead of assuming E, H in hole equal to the incident wave as in the [illegible]off method, we shall set up some integral equations to be solved for $\overline{K}, \eta, \overline{K}_m, \eta_m$ in the hole. From (11a)(9)(10) we see that $\overline{K}, \eta$ generate $\overline{E}(r), \overline{H}(r)$ that violate b.c. on screen. From (9)(14a)(14b) $\overline{K} = \frac{1}{2} \overline{n} \times \overline{H}_{0t}$, $\eta = \frac{\epsilon}{2} \overline{n} \cdot \overline{E}_{on}$ which means $\overline{K}$ and η are already known. Similarly we see that $\overline{K}_m$, η_m generate $\overline{E}(r), \overline{H}(r)$ which satisfy b.c. on screen and still are unknowns to be determined (Only continuity is required by (13e,f)).

Using only the terms which (1) are not yet fixed in the hole and (2) satisfy b.c. on screen, we retain only $\overline{K}_m$ and η_m*. (9)(10) reduce to

$$\overline{E}(r) = \frac{1}{4\pi} \int_S \overline{K}_m \times \nabla_f \phi \, dS \tag{15}$$

$$\overline{H}(r) = -\frac{1}{4\pi} \int_S \left[j\omega\epsilon \, \overline{K}_m \phi + \frac{\eta_m}{\mu} \nabla_f \phi \right] dS \tag{16}$$

S is the surface covering hole in screen

* There may be [illegible] for this step in reference (12) and (13) which I have not yet [illegible].

[illegible]

which satisfy b. c. in hole (14). [illegible]tting $\bar{E}$, and $\bar{H}$ in terms of vector and scalar potentials as is done by Schelkunoff[16]

$$\text{(a)}\quad \bar{E} = (-j\omega\mu\bar{A} - \nabla V) - (\nabla\times\bar{F})$$

$$\text{(b)}\quad \bar{H} = (\nabla\times\bar{A}) - (\nabla U + j\omega\epsilon\bar{F})$$

(1)

$$\text{(c)}\quad \bar{A} = \int_v \frac{\bar{J}e^{-jk\rho}}{4\pi\rho}\,dv \qquad \text{(d)}\quad V = \int_v \frac{q_v\, e^{-jk\rho}}{4\pi\epsilon\rho}\,dv$$

$$\text{(e)}\quad \bar{F} = \int_v \frac{\bar{M}e^{-jk\rho}}{4\pi\rho}\,dv \qquad \text{(f)}\quad U = \int_v \frac{m_v\, e^{-jk\rho}}{4\pi\mu\rho}\,dv$$

The exclusion of terms already fixed in hole and not satisfying b. c. on screen as in (15)(16) gives from (1): $\bar{M}\cdot\delta = \bar{K}_m$, $dS = \frac{dv}{\delta}$; $m_v\,\delta = \eta_m$

$$(18)\qquad \bar{E}(r) = -\nabla\times\bar{F}$$

$$(19)\qquad \bar{H}(r) = -\nabla U - j\omega\epsilon\bar{F}$$

$$(20)\qquad \bar{F}(r) = \int_S \frac{\bar{K}_m(r')\,e^{-jk|r-r'|}}{4\pi|r-r'|}\,dS(r')$$

$$(21)\qquad U(r) = \int_S \frac{\eta_m(r')\,e^{-jk|r-r'|}}{4\pi\mu|r-r'|}\,dS(r')$$

Equations (20)(21) are the integral equations to be solved for $\bar{K}_m$, η_m where $\bar{F}(r)$ and $U(r)$ are fixed by b.c. (14). A rigorous interpretation of (20) and (21) indicates that the integral is to be taken over the two boundary surfaces separated by the infinit[illegible]. By b.c. (13e) [illegible] and opposite direction of normals on left and right: $\bar{K}_m(0^-) = -\bar{K}_m(0^+)$; $\eta_m(0^-) = -\eta_m(0^+)$ This makes $\bar{F}(0) = 0$, $U(0) = 0$ from (20)(21), yet by b. c. (14) we know F(0) and U(0) are generally non-zero. This difficulty is [illegible] by taking half of F(0) and U(0) and integrating over the right surface only (i.e. solving for the diffracted wave to the right):

$$(20a)\qquad \frac{1}{2}\bar{F}(0) = \int_{+S} \frac{\bar{K}_m(r')\,e^{-jk|r-r'|}}{4\pi|r-r'|}\,dS(r')$$

(21a) $\frac{1}{2} U(0) = \int_{ns} \frac{\eta_m(r') e^{-jk|r-r'|}}{4\pi\mu |r-r'|} dS(r')$

This can be put in the form:

$$\overline{F}(r,0) = \int_{ns} \frac{\overline{E}\times\overline{n}}{2\pi} \frac{e^{-jk|r-r'|}}{|r-r'|} dS(r')$$

Bethe (1) uses this form (not used in this paper) and (2) uses $\overline{K}_m = \frac{\overline{E}\times\overline{n}}{2\pi}$ and $\eta_m = \frac{\overline{H}\cdot\overline{n}}{2\pi}$ in gaussian units. In gaussian units $\operatorname{div}\overline{H} = 4\pi\rho = 4\pi\frac{\eta_m}{\delta}$ so the above definition of η_m can be derived by reversing the sign of the fields on the left which gives $\operatorname{div}\overline{H} = \frac{2\overline{H}\cdot\overline{n}}{\delta}$ so $\eta_m = \frac{\overline{H}\cdot\overline{n}}{2\pi}$ This means that η_m in Bethe's definition includes the "magnetic charges" associated with both the diffracted waves to the left and to the right. This means that Bethe's development and this paper give consistant values of $\overline{E}\times\overline{n}$ and $\overline{H}\cdot\overline{n}$, yet Bethe's equations give twice the values of $\overline{K}_m$ and η_m defined in this paper.

$\overline{n}$ · into (6a) gives: $\nabla\cdot(\overline{E}\times\overline{n}) = -j\omega\mu(\overline{H}\cdot\overline{n})$ or

(22) $\nabla\cdot\overline{K}_m = \frac{\partial K_{mx}}{\partial x} + \frac{\partial K_{my}}{\partial y}$ $\qquad \nabla\cdot\overline{K}_m = -j\omega\eta_m$

For an infinite plane wave at an infinite plane screen $\overline{H}_o$ is constant over the hole. In actual practice and particularly for waveguides the standing wave $\overline{H}_o$, $\overline{E}_o$ will vary across the aperture. When $ka << 1$, where a is the farthest point of the aperture edge from center of gravity, taking $\overline{H}_o$ as constant is still a good approximation for simplifying $U(r,o)$ for use in (21a) to solve for η_m. Caution must be observed to avoid neglecting the curl $\overline{H}$ so that no important components of $\overline{E}$ are lost.

On the contour of the hole the b.c. $\overline{E}\times\overline{n} = 0$ on the screen requires E_{tan} vanish on the contour

(23)* $\overline{E}_c = \overline{E}\cdot\hat{c} = 0$ on contour

*This condition was not included in Bethe's original paper[1a,1b] but was published by Bethe in an M.I.T. R.L. report which remained classified for some time. Bourgin[17] in the meantime (~~published this condition~~.) said some condition like this was necessary.

The magnetic current perpendicular to contour must likewise vanish.

(24) $\overline{K}_{mp} = \overline{E}_{c} \times \overline{n} = \overline{E} \cdot (\overline{s} \times \overline{n}) = -\overline{n} \times (\overline{E} \cdot \overline{s}) = 0\,,\ K_{mp} = 0 \text{ on } C$

From (23) we have: $\oint \overline{E} \cdot d\overline{s} = 0 \qquad \text{--- (25a)}$

By Stoke's Law: $\oint \overline{E} \cdot d\overline{s} = \int_A (\nabla \times \overline{E}) \cdot \overline{n}\, da$

From (6)(8)(25a) $0 = \oint \overline{E} \cdot d\overline{s} = -j\omega\mu \int_A \overline{H} \cdot \overline{n}\, da = -j\omega \int \eta\, da$

(25b) $\int_A \eta\, da = 0$

The total "magnetic charge" or the window is zero.

4. Separation into H and E Components.

Examination of (22) indicates division of fields into two components is possible. $\overline{K}_m = \overline{K}_{mH} + K_{mE}$

$$\operatorname{div} \overline{K}_H = -j\omega \eta_m \neq 0 \qquad \operatorname{curl} \overline{K}_H \sim ka \csc\theta \operatorname{curl} \overline{K}_E \overset{*}{\to} 0$$

$$\operatorname{div} \overline{K}_E = 0 \qquad \operatorname{curl} \overline{K}_E \neq 0$$

$*\ B < \theta \leq 90^\circ,\ B \sim 20^\circ$

θ is angle of incidence (to normal of path of incident wave: $ka \ll 1$

Using "$\sim$" to mean "order of magnitude of" where $ka \ll 1$:

$$(20a)\ \tfrac{1}{2}\overline{F} = \int_{\Delta S} \frac{\overline{K}_m(r')\, dS(r')}{4\pi |r-r'|} \sim \overline{K}_m a \quad \text{---}(26a)$$

$$(21a)\ \tfrac{1}{2}U = \int_{\Delta S} \frac{\eta_m(r')\, dS(r')}{4\pi\mu |r-r'|} \sim \frac{\eta_m a}{\mu} \quad \text{---}(26b)$$

Considering the two components:

(1) H component

$$\nabla\cdot\overline{K}_m \sim K_m/a \sim \omega\eta_m \qquad (27a)$$

$$K_m \sim \omega a\, \eta_m \qquad (28a)$$

$$K_m \sim ka\sqrt{\tfrac{\mu}{\epsilon}}\,(\overline{H}\cdot\overline{n})$$

$$E_{tan}(r') \ll \sqrt{\tfrac{\mu}{\epsilon}}\, H_n(r')$$

Electric dipoles neglected ($B < \theta \leq 90^\circ$)

$$\overline{H}(r) \sim [1 + k^2a^2]\,\frac{\eta_m}{\mu}$$

$$\boxed{H(r) = -\nabla U} \qquad (29a)$$

$$E(r) \ll \sqrt{\tfrac{\mu}{\epsilon}}\, H(r)$$

Normal $\overline{H}(r)$ predominates, which is generated by incident tangential $\overline{H}(r)$.

$$\eta_m \sim \mu H_0 \qquad (30a)$$

(2) E component

$$\nabla\cdot\overline{K}_E = 0 \qquad (27b)$$

$$\eta_m = 0 \qquad (28b)$$

$$(\overline{H}\cdot\overline{n}) = 0$$

$H_n(r') = 0 \qquad E_{tan}(r')$ is present

Magnetic dipoles not present

$$H(r) = \nabla U - j\omega\epsilon\overline{F}$$

$U = 0;\quad H(r) \sim ka\sqrt{\tfrac{\epsilon}{\mu}}\, K_m$

$$\boxed{E(r) = -\nabla\times\overline{F}} \qquad (29b)$$

$$E(r) = -\nabla\times\overline{F} \sim F/a \sim K_m$$

$$\sqrt{\mu/\epsilon}\, H(r) \ll E(r)$$

Tangential $\overline{E}(r)$ predominates, which is generated by incident normal $\overline{E}(r)$.

$$K_m \sim E_0 \qquad (30b)$$

5. Solution for the H component.

We neglect terms of order ka since in this approximation $ka \ll 1$.

For uniform incident field by b.c. (14a) and (29a) we have at z = 0:

(31) $\bar{H} = \frac{1}{2}\bar{H}_{oy} = -\nabla U = -\bar{a}_y \frac{\partial U}{\partial y}$ $\quad U = -\int \frac{H_{oy}}{2}\, dy = -\frac{1}{2} H_o y = -\frac{1}{2}\bar{H}_o \cdot \bar{r}$

(31) in (26b) gives integral equation for η_m:

(32) $$\boxed{-\frac{1}{4}\bar{H}_o \cdot \bar{r} = \int_{hs} \frac{\eta_m(r')}{4\pi\mu |r-r'|}\, dS(r')}$$

The magnetic moment is:

(33) $\bar{\chi} = \int_{hs} \eta_m(r')\, \bar{r}'\, dS(r')$

$\bar{r}' = \bar{a}_x x' + \bar{a}_y y'$

Using ~~$\bar{r}' = \bar{a}_x x' + \bar{a}_y y'$~~ and (22) in (33).

$\bar{\chi} = \bar{a}_x \chi_x + \bar{a}_y \chi_y = \bar{a}_x \int_{hs} \eta_m(r') x'\, dS + \bar{a}_y \int_{hs} \eta_m(r') y'\, dS$

$-j\omega \chi_x = \int_{hs} x (\nabla \cdot \bar{K}_m) dS \qquad -j\omega \chi_y = \int_{hs} y (\nabla \cdot \bar{K}_m) dS$

$-j\omega \chi_x = \int x (\nabla \cdot \bar{K}_m) dS = \int \nabla \cdot (x \bar{K}_m) dS - \int \bar{K}_m \cdot (\nabla x) dS \quad \cdots (35)$

1st term by Gauss Law and (24): $\int_{hs} \nabla \cdot (x\bar{K}_m) dS = \oint (x\bar{K}_m)_p\, dl = 0$

$\therefore \quad \boxed{j\omega \bar{\chi} = \int \bar{K}_m\, dS} \cdots (36) \quad$ or $\quad \boxed{j\omega \bar{\chi} = -\bar{n} \times \int \bar{E}\, dS} \quad (36a)$

By (36) $\bar{\chi}$ is a magnetic dipole moment in the plane of the screen. It is known, if E_{tan} over aperture is known. (As will be shown in the next section, the electric dipole and hence the complete solution is determined by E_{tan} over the aperture.)

Examination of (32)(33) shows that η_m and $\bar{\chi}$ are linear functions of H_{ox} and H_{oy}, so we write $\bar{\chi}$ in terms of M_{ij}, the components of a magnetic polarizability tensor.

$$X_x = \hat{M}_{xx} H_{ox} + \hat{M}_{xy} H_{oy}$$

(3) $$X_y = \hat{M}_{yx} H_{ox} + \hat{M}_{yy} H_{oy}$$

The originally incorrect sign of M was corrected by Bethe[18]. The sign can be found from the following development related to Stratton's discussion of Poynting's Theorem[19]. The stored energy is:

(38a) $$W = \frac{1}{2} \int_v [\bar{E} \cdot \bar{D} + \bar{H} \cdot \bar{B}]\, dv \qquad /\text{inst}$$

In the equations marked "/inst" the field vectors $\bar{E}, \bar{D}, \bar{H}, \bar{B}$ are the instantaneous real values. In general in this paper, the field vectors indicate the peak values (or the field vectors with the time factor $e^{j\omega t}$ omitted).

Eq (38a) is also valid where $\bar{E}, \bar{D}, \bar{H}, \bar{B}$ are average values.

Substituting (18)(19)(6a)(6c) into (38a)

(39a) $$W_{nv} = \frac{1}{2}\frac{1}{2} \int_v \left\{ \begin{array}{l} -\epsilon \nabla \cdot (\bar{F} \times \bar{E}) + j\omega\mu\epsilon\, \bar{H} \cdot \bar{F} + \epsilon\, \bar{F} \cdot \bar{J}_m \\ -\mu \nabla \cdot (\bar{H} U) - j\omega\mu\epsilon\, \bar{H} \cdot \bar{F} + \hat{U} \rho_m \end{array} \right\} dv \qquad /a$$

(39b) $$\frac{1}{2} W_{nv} = \frac{1}{2}\frac{1}{2} \int_{nv} (\epsilon\, \bar{J}_m \cdot \bar{F} + U \rho_m)\, dv + \frac{1}{4} \int_{S} (\bar{F} \times \bar{E} + \hat{U} \bar{H}) \cdot d\bar{s} \qquad /a$$

The second 1/2 comes from division at z = 0 as in (20a)(21a). The volume is bounded by a hemisphere on the right and the conducting surface of the screen plus a surface through aperture. Since the waves are propagated by a finite velocity, we may take the radius of the hemisphere sufficiently large so that the diffracted wave has not reached the surface yet. Then the second integral is zero over the hemisphere. The surface integral over the screen is zero since $\bar{F} \times \bar{E} \cdot \bar{n} = 0$ and $U \bar{H} \cdot \bar{n} = 0$ on screen. This leaves

(39c) $$\frac{1}{2} W_{nv} = \frac{1}{2} \int_{nv} (\epsilon\, \bar{J}_m \cdot \bar{F} + U \rho_m)\, dv \qquad \text{approx}$$

$$W_{nv} = \frac{1}{2} \int_{nv} (\epsilon\, \bar{J}_m \cdot \bar{F} + U \rho_m)\, dv$$

change from instantaneous $\hat{\bar{\bar{I}}}_m, \hat{F}, \hat{U}, \hat{\eta}_m$ to peak values.

(40) $$\boxed{\tfrac{1}{2}W_\omega = \frac{1}{8}\int_{As}\left[U\eta_m + \epsilon\,\bar{F}\cdot\bar{K}_m\right]dS}$$ /peak

For H component by (29a) second term is of order $(ka)^2$, $ka \ll 1$ so by (21a)

(40a) $$W_\omega \cong \frac{1}{4}\cdot\frac{1}{2}\int_{As} U\eta_m\,dS = \frac{1}{8\pi\mu}\iint \frac{\eta_m(r)\,\eta_m(r')}{|r-r'|}\,dS(r)\,dS(r') > 0$$

(40a) is positive since it is a self potential. Using (40a)(32)(33):

$$W_\omega = -\frac{1}{8}\frac{1}{\mu}\int \bar{H}_0\cdot\bar{r}\,\eta_m\,dS = -\frac{\bar{H}_0}{8\mu}\cdot\int \bar{r}\,\eta_m\,dS = -\frac{\bar{H}_0\cdot\bar{\chi}}{8\mu} > 0$$

So $\bar{H}_0$ and $\bar{\chi}$ have opposite signs

(41) $$W_\omega = -\frac{1}{8\mu}\bar{H}_0\cdot\bar{\chi} > 0$$

Let $M_{ij} = -\hat{M}_{ij}$ in (37) then $M_{ij} > 0$

(41a) $$\begin{cases} \chi_x = -M_{xx}H_{ox} - M_{xy}H_{oy} \\ \chi_y = -M_{yx}H_{ox} - M_{yy}H_{oy} \end{cases}$$

Bethe points out that the energy relation (41) is analogous to that of the ordinary theory of magnetism: "Therefore, just as we can conclude that the tensor of magnetic permeability is symmetrical, so we believe that in our case we can conclude that the M-tensor is symmetrical."20

(42) $$M_{yx} = M_{xy}$$

We need only assume that $\bar{\chi}$ is the derivative with respect to $\bar{H}_o$ of:

(42a) $$8W = \tfrac{1}{2}M_{xx}H_{ox}^2 + \tfrac{1}{2}(M_{xy}+M_{yx})H_{ox}H_{oy} + \tfrac{1}{2}M_{yy}H_{oy}^2$$

If we accept (42) the tensor M can be transformed to principal axes. For non-symmetrical apertures the directions of the principal axes must be determined from the integral equation.

Let $\bar{\ell}$ and $\bar{m}$ be unit vectors in the directions of the two principal axes, and $H_{o\ell}$, H_{om} be respective components of H_o. Then in terms of principal axes, the magnetic moment is:

$$\boxed{\bar{X}_{(dipole)} = -M_1 H_{o\ell}\,\bar{\ell} - M_2 H_{om}\,\bar{m}} \tag{43}$$

M_1, M_2 are the principal magnetic polarizabilities in units of permeability times volume $\left(M \sim \mu a^3 \text{ henry-meter}^2 \right)$

An alternate form from (36b) is:

$$\bar{n} \times \int \bar{E}\, dS = +j\omega \left(M_1 H_{o\ell}\,\bar{\ell} + M_2 H_{om}\,\bar{m} \right) \tag{44}$$

6. Solution for E component.

For the electric component we have from (22)(2 b):

(45) $$\eta_m = \frac{+j}{\omega}\,\nabla\cdot\overline{K}_m = \frac{j}{\omega}\,\nabla\cdot(\nabla\times\overline{E}) = 0$$

This means that lines of magnetic current must be closed or that E_{tan} can be derived from a potential function:

(46) $$\overline{E}_t = -\nabla\phi$$

(45) and (46) are true irrespective of size of hole. By b.c. (23):

ϕ = constant on the contour of aperture. The constant may be taken as zero without changing E_t from (46): $\phi = 0$ on contour (46a)

(42)(44a) $$-\nabla\times\overline{F} = \tfrac{1}{2}\,\overline{E}_{on} \qquad -(\nabla\times\overline{F})\times\overline{r} = \tfrac{1}{2}\,\overline{E}_{on}\times\overline{r}$$

$$(\nabla\times\overline{F})\times\overline{r} = \overline{F}(\nabla\cdot\overline{r}) - \overline{r}(\nabla\cdot\overline{F}) \qquad \nabla\cdot\overline{F} = 0,\ \nabla\cdot\overline{r} = 2\ \text{[illegible]}$$

$$-2\overline{F} = \tfrac{1}{2}\,\overline{E}_{on}\times\overline{r} \qquad \overline{F} = -\tfrac{1}{4}\,\overline{E}_{on}\times\overline{r} \qquad (46b)$$

Now (46b) and (26a) [illegible] give the Integral equation to be solved – $\overline{K}_m$

$$\boxed{-\tfrac{1}{4}\,\overline{E}_{o,n}\times\overline{r} = \int_{ns}\frac{\overline{K}_{m}(r')}{4\pi|r-r'|}\,dS(r')} \qquad (46c)$$

Given K_{m_t} [illegible] on the [illegible] boundary [illegible] the [illegible], we may [illegible] the [illegible] Hertz vector [illegible] see [illegible] components [illegible] by [illegible]ities. This [illegible] to that of [illegible] is used in place of Π since [illegible] current [illegible] electric currents [illegible] function, [illegible] Π [illegible], [illegible] to [illegible] eq. (10)

(4)a. $$\nabla^2\,\overline{\Pi}_j + \omega^2\mu\epsilon\,\overline{\Pi}_j = -M_j = -\frac{X_j}{\mu}$$

The dipole magnetization $\overline{X}$ is used here and everywhere in this paper instead of loop magnetization $\overline{M}$ to make the inhomogeneous wave equations of identical form for both magnetic and electric currents.

(47b) $\overline{X}_o = \overline{\mathcal{K}} = \frac{-j\,\overline{J}m}{\omega} = \frac{-j\,\overline{K}m}{\omega\,\delta}$

corresponding to Stratton, eq. [illegible]

(47c) $\overline{\Pi}^{*} = \frac{\epsilon}{4\pi} \int \overline{X}_o(\xi)\, \frac{e^{-jkr}}{r}\, dv$

The total field is $\overline{\Pi}^{*} = \sum_{n=0}^{\infty} \overline{\Pi}^{*(n)}$ where $\overline{\Pi}^{(0)}$ is magnetic [illegible] represented by a sum [illegible] $\overline{\Pi}^{(1)}$ [illegible] quadrupole, [illegible] dipole [illegible])

Then substituting $\overline{X}$ for $\overline{P}$ [illegible] p. 232, [illegible]

(47d) $\overline{\Pi}^{*(1)} = \frac{1}{4\pi\epsilon} \overline{X}^{(1)} \frac{e^{j(\omega t - kr)}}{r}$ $\qquad \overline{X}^{(1)} = \int_{ns} \overline{X}_o\, dS$ (47e)

From [illegible] $\overline{X}_o = [illegible] = \frac{-j\,\overline{K}m}{\omega} = -\frac{j}{\omega}(\overline{a}_x\, [illegible] - \overline{a}_y\, [illegible])$ (47f)

[illegible] (36b) [illegible]

$\overline{X}^{(1)} = [illegible] \int \overline{E}\, [illegible]$ $\qquad \int \overline{E}_t\, dS = -\int \nabla \phi\, dS = -\oint \phi\, \overline{a}_n\, ds$

$\int \overline{E}_t\, [illegible] = -\oint \phi\, \overline{n} \times d\overline{s}$ $\qquad \overline{X}^{(1)} = 0$ (47g)

[illegible]

[illegible]

$[illegible] \frac{+jk}{4\pi\epsilon}\left(\frac{1}{R} - \frac{j}{kR^2}\right) e^{+j(\omega t - kR)} \int \overline{X}_o(\xi)\, R_1 \cos\gamma\, dv$ (48

[illegible]

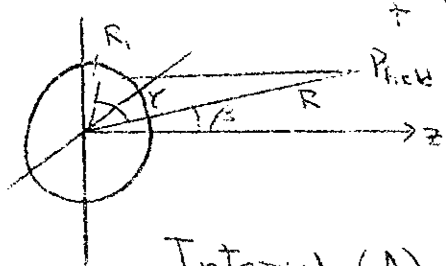
$\overline{X}\, [illegible] \cos\gamma = R_1 \cdot \overline{R}\, \frac{\overline{X}_o}{R} = \frac{1}{2R}\left[(\overline{R}_1 \times \overline{X}_o) \times \overline{R} + \overline{X}_o(\overline{R}_1 \cdot \overline{R}) + \overline{R}_1(\overline{R} \cdot \overline{X}_o)\right]$

(48a)

Changing to surface integral, the integral part of (49a) becomes

$$\int_{ns} \overline{X}_o R_1 \cos\gamma \, dS = \underbrace{\int_{ns} \frac{(\overline{R}_1 \times \overline{X}_o) \times \overline{R}}{2R} dS}_{(A)} + \underbrace{\int_{ns} \frac{\overline{X}_o (\overline{R}_1 \cdot \overline{R})}{2R} dS}_{(B)}$$

$$+ \underbrace{\int_{ns} \frac{\overline{R}_1 (\overline{R}_1 \cdot \overline{X}_o)}{2R} dS}_{(C)} \qquad \text{---- (49c)}$$



Integral (A) of (49c):

$$\int_{ns} \frac{(\overline{R}_1 \times \overline{X}_o) \times \overline{R}}{2R} dS = -\frac{j}{\omega} \int_{ns} \frac{E_x x + E_y y}{2} dx\,dy \; \overline{a}_z \times \left(\frac{\overline{R}}{R}\right) -$$

$$\int E_x x \, dS = -\int \frac{\partial \phi}{\partial x} dS = -\int \left[\frac{\partial}{\partial x}(\phi x) - \phi\right] dS$$

$$= +\int \phi \, dS - \oint (\phi x) \cos(a_n, x) \, ds$$

by (46e) $\quad \int E_x x \, dS = \int E_y y \, dS = \int \phi \, dS \qquad$ --- (49d)

Integral (B) of (49c).

$$\int_{ns} \frac{\overline{X}_o (\overline{R}_1 \cdot \overline{R})}{2R} dS = -\frac{j}{\omega} \int (\overline{a}_x E_y - \overline{a}_y E_x) \frac{(x X + y Y)}{2R} dx\,dy = -\frac{j}{2\omega} \left(\overline{a}_x \frac{Y}{R} - \overline{a}_y \frac{X}{R}\right) \text{[illegible]}$$

Integral (C) of (49c)

$$\int_{ns} \frac{\overline{R}_1 (\overline{R} \cdot \overline{X}_o)}{2R} dS = -\frac{j}{\omega} \int_{ns} \left(\text{[illegible]}\right)\left(\frac{X}{R} E_y - \frac{Y}{R} E_x\right) dS = \frac{+j}{2\omega} \left(\overline{a}_x \frac{Y}{R} - \overline{a}_y \frac{X}{R}\right) \text{[illegible]}$$

By (49a)(49c) integrals (B) and (C) cancel leaving (A). Putting (49d)

(49e) back into (4[illegible]) we have:

$$(49e) \quad \overline{\Pi}^{(1)} = \frac{1}{4\pi} \text{[illegible]} \left[\overline{a}_z \int \phi \, dS\right] \times \nabla R \left(\frac{1}{R} - \text{[illegible]}\right) e^{j(\omega t - kR)} \; ; \quad \text{[illegible]} = \left(\frac{\overline{R}}{R}\right)$$

Comparing (49e) with Stratton, p. 434, eq. (21), $\overline{\Pi}^{(0)} \rightarrow$ [illegible] $\overline{\Pi}^{(1)}$

we see that $\overline{\Pi}^{(1)}$ is the contribution of an electric dipole of moment

$$(49f) \qquad \overline{p}^{(1)} = \overline{a}_z \, \epsilon \int \phi \, dS$$

Since the other integrals cancelled there is no magnetic quadrupole moment from the [illegible] component. By (49e) and (49f).

(50) $\boxed{\overline{p}^{(1)} = \overline{a}_z \frac{\epsilon}{2} \int_{ns} \overline{E} \cdot \overline{r}' \, dS(r')}$ ($\overline{E}_r$ comes from solution of (46a))

In §4 it was shown that the component is generated by the incident normal $E(r)$. Consequently we take p proportional to normal incident $\overline{E}$.

(51) $\boxed{\overline{p} = \overline{n} \; P(\overline{E}_o \cdot \overline{n})}$

P is the electric polarizability of the aperture. $P \sim \epsilon a^3$ (farad-meter^2)

Alternately (50) can be obtained from $\overline{K}_m$ directly. Examining $\overline{m} = \frac{\overline{X}}{\mu} = \frac{1}{2}\int_v \overline{r} \times \overline{J} \, dv$ in Stratt p22 and noting sign difference from (6a,b):

(50a) $\overline{p}/_\epsilon = \frac{1}{2} \int \overline{K}_m \times \overline{r} \, dS$

(50b) $\overline{p} = \frac{\epsilon}{2} \int (\overline{E} \times \overline{n} \times \overline{r}) \, dS = \frac{\epsilon}{2} \int [\overline{n}(\overline{E} \cdot \overline{r}) - \overline{r}(\overline{E} \cdot \overline{n})] \, dS = \overline{a}_z \frac{\epsilon}{2} \int_{ns} \overline{E} \cdot \overline{r}' \, dS$

Note (50b) is same as (50). To determine sign of P in (51) we use (40,43)

(52) $\frac{1}{2} \delta W_{rr} = \int_{ns} \epsilon \, \overline{F} \cdot \overline{K}_m \, dS$

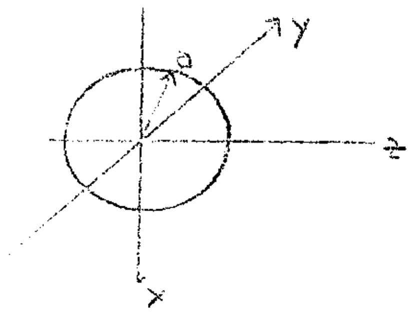
(53) from (46a)(52) $\frac{1}{2} W_{rr} = \frac{\epsilon}{16\pi} \iint \frac{\overline{K}_m(r) \cdot \overline{K}_m(r')}{|r - r'|} \, dS(r) \, dS(r') > 0$

From (53)(52)(48) $\frac{1}{2} \delta W_{rr} = \int_{ns} \epsilon(-\frac{1}{4}\overline{E}_{on} \times \overline{r}) \cdot \overline{K}_m dS = \frac{\epsilon}{4} \overline{E}_{on} \cdot \int \overline{K}_m \times \overline{r} \, dS$

$$\frac{1}{2} W_{rr} = \frac{\epsilon}{32} \overline{E}_{on} \cdot \left(\frac{2\overline{p}}{\epsilon}\right) = \frac{\overline{E}_o \cdot \overline{n}}{16} P(\overline{E}_o \cdot \overline{n}) = \frac{1}{16} P(\overline{E}_o \cdot \overline{n})^2 > 0 \quad (54)$$

Since $(\overline{E} \cdot \overline{n})^2$ is positive, P is positive. The difference in sign between M and P is not due to their being magnetic and electric, but due to being parallel and perpendicular to screen

Example of Circular Aperture.



Bethe[1] has obtained a solution for a circular aperture of radius a. Bouwkamp[3] has corrected Bethe's solution which gave incorrect E_{tan} for H component near aperture and found Bethe's radiation field results to be correct. Bouwkamp has also obtained a series solution of which the first term corresponds to Bethe's approximation developed here.

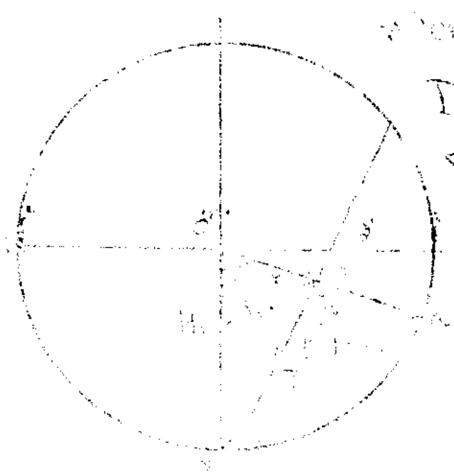
a. H Component. (Corresponds to TE mode circular iris of Slater[24])

The integral equation is (32):

$$(55)(32) \qquad -\frac{1}{4}\,\overline{H}_0 \cdot \overline{r} = \int \frac{\eta_m(r')}{4\pi\mu|\overline{r}-\overline{r}'|}\, dS(r')$$

Since Bouwkamp's correct solution in oblate spheroidal coordinates has not yet been published (to appear in second of two papers), we will outline Bethe's procedure and then give the correct solution given without formal development in Bouwkamp's first paper.

Bethe states that a constant magnetic field is produced by a uniform distribution of dipoles in an ellipsoid as is developed by Stratton[23]. [illegible] consider the aperture to be filled by an ellipsoid of semi-axes a, [illegible].



[illegible] $h \to 0$ $\quad \frac{r^2}{a^2} + \frac{z^2}{h^2} = 1 \quad z \cong (a^2 - r^2)^{1/2}$

Dipole surface density $\quad \overline{\chi} = \alpha \overline{H}_0 \quad$ [illegible]

Surface charge density $\quad \eta_m = \nabla \cdot \overline{\chi}$ [illegible]

$\operatorname{div} \overline{J} A_y = \overline{J} \cdot \operatorname{grad} A_y \quad$ for $\overline{A} = $ [illegible]

$\eta_m = \overline{H}_0 \cdot \alpha$ [illegible] $\qquad$ (56)

[illegible] $= \frac{\overline{r}'}{(\ldots)}$ [illegible] $\eta_m = \alpha \frac{\overline{H}_0 \cdot \overline{r}'}{\sqrt{a^2 - r^2}}$ [illegible]

$$H_0 \cdot r' = H_0 r + H_0 \rho \cos(\phi-\beta) \quad \text{--- (56b)}$$

(56b) in (55)

$$-\tfrac{1}{4}\,\overline{H}_0\cdot\overline{r} = \alpha \int_0^{2\pi} \frac{d\beta}{4\pi\mu} \int_0^{\rho} \frac{H_0 r + H_0 \rho\cos(\phi-\beta)}{(a^2-r'^2)^{1/2}\,\rho}\,\rho\, d\rho \quad \text{--- (57)}$$

$\hat{x} = \rho + r\cos\beta$, $\rho = \hat{x} - r\cos\beta$, $d\rho = d\hat{x}$. Combining RP, PS makes $2\pi \to \pi$

$$-\tfrac{1}{4}\,\overline{H}_0\cdot\overline{r} = \alpha \int_0^{\pi} \frac{d\beta}{4\pi\mu} \left\{ \overline{H}_0\cdot\overline{r} \underbrace{\int_{-s}^{+s} \frac{ds}{(a^2 \ldots)^{1/2}}}_{(\mathrm{I})} + H_0\cos(\phi-\beta) \underbrace{\int_{-s}^{+s} \frac{\hat{x}\, d\hat{x}}{(a^2-r'^2)^{1/2}}}_{(\mathrm{II})} - H_0\cos(\phi-\beta) \underbrace{\int_{-s}^{+s} \frac{r\cos\beta\, d\hat{x}}{(a^2 \ldots)}}_{(\mathrm{III})} \right.$$

--- (57)

Integrals I, II are π, $-\pi r\cos\beta$ by Ref. (27); III is zero by Ref. (28)

$$-\tfrac{1}{4}\,\overline{H}_0\cdot\overline{r} = \frac{\alpha}{4\mu}\int_0^{\pi} \left(\overline{H}_0\cdot\overline{r} - H_0 r\cos\beta\cos(\phi-\beta)\right) d\beta = \frac{\alpha\pi}{4\mu}\left(\tfrac{1}{2}\,\overline{H}_0\cdot\overline{r}\right)$$

$$\alpha = \frac{2\mu}{\pi} \quad \text{--- (57)}$$

(56a)(57)

$$\boxed{\eta_m = -\frac{2\mu_0}{\pi}\,\frac{\overline{H}_0\cdot\overline{r}'}{(a^2-r'^2)^{1/2}}} \quad \text{--- (58)} \quad \left(\frac{\text{webers}}{\text{meter}^2}\right)$$

To obtain the correct value for $\overline{K}_m$, we have to go back to (2[illegible]). Bethe calculated $\overline{K}_m$ from (22) [illegible] and thereby missed [illegible] of $\overline{K}_{m_H}$ which caused $\overline{K}_m$ to fail [illegible] b.c. (24) $\overline{K}_m = 0$ on [illegible].

Bouwkamp[3] has obtained the following result as the first term in power series in (ka) in [illegible] coordinates:

(58) $$\boxed{\overline{K}_{m_H} = [\text{illegible}] \frac{H_0}{[\ldots]\,x^2 \ldots} + [\ldots]\,\frac{[\ldots]}{3\pi}\,\frac{(2a^2 \ldots 2\,y^2)H_0}{\sqrt{a^2 \ldots}\; x^2 - y^2}}$$

$$- [\text{illegible}] \left\{ \frac{[\ldots] - (2a^2 - 2x^2 - 2y^2)\,y}{(a^2 - x^2 - y^2)^{3/2}} \right\} H_0 = -\frac{\mu}{\pi}\,\frac{[\ldots]\,y}{a^2 - x^2 - y^2}\,\eta$$

Thus (58) agrees with ([illegible] c).

b. <u>Electric [illegible]</u>. [illegible] circular [illegible]

Bouwkamp did not [illegible] component in his first term. Since Bethe's [illegible] of [illegible] by Bouwkamp [illegible] for [illegible]

[illegible]

From (46c) the integral equation is:

(59)(46c) $$-\frac{1}{8}\overline{E}_{0m} \times \overline{r} = \int_{D_1} \frac{\overline{K}_{mE}(r')}{4\pi|r-r'|}\, dS(r')$$

(59a) $$-\frac{1}{8}E_{0y} = \int_{D_1} \frac{K_y\, dS}{4\pi|r-r'|}$$

(55) $$-\tfrac{1}{4}H_{0y} = \int_{D_1} \frac{\eta_m/\mu\, dS}{4\pi|r-r'|}$$

Comparing components with (55) [illegible] see that they are the same equations with different symbols, so we can use (57d) which gives by comparison:

(60) $$\boxed{\overline{K}_{mE} = \frac{1}{\pi(a^2-r'^2)^{1/2}}\,\overline{r}' \times \overline{E}_0}$$ (volts)

c. Magnetic Polarizability.

(61) $$\overline{X} = -\tfrac{1}{\omega}\int(\overline{K}_{mH} + \overline{K}_{mE})\,dS$$ — from [illegible]

The x-component of $\overline{K}_m$ and the [illegible] $\overline{K}_{mE}$ part will both integrate to zero.

It is simpler to use (33) and (57d)

$$\overline{X} = \int_{D_1} \gamma_m(r)\, r'\, dS(r') = \overline{a}_x \int \eta_m x\, dS + \overline{a}_y \int \eta_m y\, dS = \overline{a}_y\left(\frac{-4/\mu}{\pi}\right)\int_0^{2\pi}\int_0^a \frac{r' \cdot H\, r'^2\cos^2\phi\, d\ldots}{(a^2-r'^2)^{1/2}}$$

(62) $$\overline{X} = -\overline{a}_y \frac{4\mu H}{\pi}\int \frac{\ldots}{(a^2-r'^2)}\, d(r') = -\overline{a}_y \frac{4}{3}a^3\mu H_0 = \frac{4}{3}a^3\mu H_0$$

(63) $$M_{yy} = -\frac{X_y}{H_{0y}} = \boxed{\frac{4}{3}a^3\mu = M_{yy}}$$

d. Electric Polarizability.

$$\overline{p} = \epsilon\int \overline{E}\cdot r'\, dS = \epsilon\int(\overline{n}\times\overline{K}_{mE})\cdot\overline{r}'\, dS = \frac{\epsilon}{\pi}\int\frac{(\overline{n}\times\overline{r}'\times\overline{E}_0)\cdot\overline{r}'}{\ldots}$$

$$(\overline{n}\times r'\times\overline{E}_0)\cdot r' \ldots = \frac{\epsilon E_0}{2\pi}\int\ldots = \frac{\epsilon E_0}{2}\,\frac{4}{3}a^3$$

$$\overline{p} = +\frac{2a^3}{3}\epsilon E_0 \qquad (64) \quad (51)(61) \qquad P = \frac{\overline{p}}{E_0} = \boxed{+\frac{2a^3}{3}\epsilon = P}$$ (65)

8. Summary of Results for Various [illegible]apes.

a. Circular Aperture. [illegible]dius a

Bethe's Gaussian System

$$\bar{K} = \frac{\bar{E} \times \bar{n}}{2\pi} \quad \text{and} \quad \eta_m = \frac{\bar{H} \cdot \bar{n}}{2\pi}$$

$$M_1' = M_2' = \frac{4}{3} a^3 \qquad \text{[illegible]}^*$$

$$P' = \frac{2}{3} a^3 \qquad (71)$$

$$\bar{\mathcal{X}}' = -\frac{M_1'}{2\pi} \bar{H}_0' \ (\text{emu})$$

M.K.S. from §7 this paper:

$$\mu = \mu'/\mu_0 \qquad \mu_0 = 4\pi \times 10^{-7}$$

$$\frac{M_1}{\mu} = \frac{M_2}{\mu} = \frac{4}{3} a^3 \qquad (80)^+$$

$$\frac{P}{\epsilon} = \frac{2}{3} a^3 \qquad (81)$$

$$\bar{\mathcal{X}} = -M_1 H_0 \ (\text{[illegible]})$$

\+ These two different methods of [illegible]viding the problem into left and right halves are consistent as far as the [illegible]olution for $\bar{E} \times \bar{n}$ and $\bar{H} \cdot \bar{n}$ in the aperture. The definition by Bethe of $\eta_m = 2\left(\frac{\bar{H} \cdot \bar{n}}{4\pi}\right)$ appears to [illegible] factor of 2 into the magnetic [illegible] moments.

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* From [illegible]ensional analysis one [illegible]: $\frac{M}{\mu} = \frac{4\pi}{2\pi} M' = 2$. The difference in [illegible] might resolve this [illegible] $\left(\frac{M}{M'\mu} = 1, \text{ not } 2.\right)$ in that:

Bethe apparently split[+] the pro[illegible]m [illegible] symmetrical [illegible] H_{2tan} by [illegible] by using $\eta_m = \frac{\bar{H} \cdot \bar{n}}{2\pi}$ not $\eta_m = \frac{\bar{H} \cdot \bar{n}}{4\pi}$ [illegible] paper the equivalent [illegible] by 2 at [illegible].

Checking against results for [illegible] of [illegible] around [illegible] Waveguide Handbook[23] confirms $\frac{M}{M'\mu} = 1$ [illegible] results for other [illegible] transposed from Bethe's Gaussian system [illegible] development [illegible] for the circular [illegible].

b. [illegible]

[illegible]