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CALCULATION OF ΔF AND Q_c BY USE OF A STATIONARY INTEGRAL EQUATION FOR AN IRIS-COUPLED RESONATOR.

by

F. B. WOOD

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Advanced Development Laboratory
International Business Machines Corporation
San Jose, California

Calculation of Δ f and Qc by Use of a Stationary Integral Equation for an Iris-Coupled Resonator.*

F. B. Wood**

Summary - Iris coupling between waveguides and resonators is investigated. In the case of high Q resonators common to microwave frequencies, the dyadic Green's functions are modified by the method of perturbation of boundary conditions to account for the wall losses.

The integral equation for the input admittance of a cavity resonator coupled to a waveguide by means of an iris is found to have each admittance component in the variational form of J. S. Schwinger. The square of the transformer ratio of the equivalent circuit is found to be stationary when the waveguide and resonator fields have a similar form. The integral equation for the admittance leads directly to a simple equivalent circuit.

The coupled Q of the resonator is obtained by inserting an approximating trial field in the integral equation, and then taking the derivative of the input admittance with respect to frequency. The dimensions of a sample rectangular resonator are chosen so that the susceptance values calculated by Marcuvitz for rectangular irises in infinite waveguides can be used. Bethe's system of lumped constants for small irises is extended to a system of "v-factor" and "q-factor" curves for computing Δ f and Qc of a class of resonators of different modes and sizes where the iris field is approximately the same for each resonator.

Curves of Δ f and Qc are calculated for a $TE_{1,0,1}$ rectangular resonator, and are then transformed to approximation curves for a $TEM_{0,0,3}$ coaxial resonator. Direct calculations of Qc are made from the integral equation for the coaxial resonator for a part of the range of iris sizes. The two sets of theoretical curves are compared with experimentally observed curves of Qc.

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**Now at Advanced Development Laboratory, International Business Machines Corporation, San Jose, California.

INTRODUCTION

In general it has not been practical to obtain exact solutions for the input impedance of a cavity resonator with finite wall conductivity. The standard practice is to obtain a solution for infinitely conducting walls and insert a term at some stage of the development to account for the wall losses. E. U. Condon [1] put a damping term in the differential equation for a mode of oscillation of a probe coupled cavity. In volume 8 of the Radiation Laboratory Series, two methods were used: (1) Extension of Fosters theorem for slightly lossy networks by introducing a complex frequency [2], and (2) Adding a damping term in the dynamical equation for a loop coupled resonator in the Lagrargian treatment developed by Crout and Banos[3]. Slater[4] develops the input impedance for a lossless resonator and inserts the wall loss damping term by analogy with an equivalent circuit.

In this analysis, the same terms for wall losses which are derived by the above investigators are developed by a different technique. Here the admittance for an iris-coupled resonator is obtained by the development of a modified Green's function through the method of perturbation of the boundary conditions. The advantages of this method are: (1) The perturbation of boundary conditions brings in the principal wall loss terms in the admittance formula directly without recourse to analogy; (2) the use of the modified dyadic Green's functions permits a simple and direct development of the admittance as an integral equation in stationary form suitable for use with trial iris fields, and (3) a simple equivalent circuit comes directly from the integral equation.

1. Condon, E. U., "Forced Oscillation in Cavity Resonators", Jour. Appl. Phys., Vol. 12, Feb. 1941, pp 129-132.

2. Montgomery, C.G., "Principles of Microwave Circuits", M.I.T. Radiation Laboratory Series, Vol. 8, McGraw-Hill (1948), pp 215-218.

3. Montgomery, C.G., loc. cit., pp 218-225.

4. Slater, J.C., "Microwave Electronics", N.Y. Von Nostrand (1950), pp 48, 50, 69, 75.

GREEN'S FUNCTIONS

Dyadic Green's Functions

In the solution of the iris coupling problem, the basic procedure is to obtain an equation for the electromagnetic field in the resonator in terms of the field on the boundary. From this, an integral equation is obtained for the input admittance as a function of frequency.

Given a region V bounded by a surface S, it is desired that the electromagnetic field in the region be obtained from a knowledge of the current distribution in the volume and on the surface. Since the currents and their related fields are both vectors, it can be seen that a Green's function in its most general form would have to be a tensor or dyadic which transforms an element in current vector space to an element in field vector space. Schwinger[5] has defined electric and magnetic dyadic Green's functions $\overleftrightarrow{\Gamma}^{(1)}(r, s)$ and $\overleftrightarrow{\Gamma}^{(2)}(r, s)$ for a region bounded by a surface so that they give the electric and magnetic fields respectively due to electric and magnetic current distribution J(r) and Jm(r) as follows where n is the unit normal into the region:

$$\bar{E}''(r) = \int_V \overleftrightarrow{\Gamma}^{(1)}(r, s) \cdot \bar{J}(s) \, dV(s) \quad , \qquad (1)$$

$$\bar{H}''(r) = \int_V \overleftrightarrow{\Gamma}^{(2)}(r, s) \cdot \bar{J}_m(s) \, dV(s). \qquad (2)$$

By deriving the wave equations for E(r) and H(r) from Maxwell's equations, and utilizing the above definitions, a pair of differential equations[6] can be obtained for these Green's functions.

5. Schwinger, J. S. "Theory of Obstacles in Resonant Cavities and Waveguides", M. I. T. Radiation Laboratory Report 205 (43-34), May 21, 1943 (PB 2859).

6. Levine, H. and Schwinger, J. "On the Theory of Electromagnetic Wave Diffraction by an Aperture in an Infinite Plane Conducting Screen", Communications on Pure and Applied Math, Vol III, No. 4, Dec. 1950, pp 355-391.
These differential equations are given by equations (3. 7) and (3. 11). To change to M. K. S. units, multiply the right hand side of each respectively by $-j\omega\mu$ and $-j\omega\epsilon$

Expansion of Green's Functions in Normal Modes

The Green's functions can be expanded in normal modes in an interior region bounded by infinitely conducting walls. The boundary conditions on the electromagnetic field and the Green's functions are then

$$\bar{n} \times \bar{E}(r) = 0, \qquad \bar{n} \times \overleftrightarrow{\Gamma}^{(1)}(r,s) = 0, \qquad (3)$$

$$\bar{n} \times \text{curl}\ \bar{H}(r) = 0, \qquad \bar{n} \times \nabla \times \overleftrightarrow{\Gamma}^{(2)}(r,s) = 0. \qquad (4)$$

Substituting[7] $\bar{H}(r)$ and $\overleftrightarrow{\Gamma}^{(2)}(r,s) \cdot \bar{e}$, where $\bar{e}$ is an arbitrary vector, into Green's second vector identity and utilizing the transpose relationships between the two Green's functions, and applying the boundary conditions of equations (3) and (4) result in the following equation for $\bar{H}(r)$ in terms of tangential $\bar{E}$ on the boundary:

$$\bar{H}(s) = + \int_s \left[\bar{E}(r) \times \bar{n}\right] \cdot \overleftrightarrow{\Gamma}^{(2)}(r,s)\, dS(r). \qquad (5)$$

The general expansion of the dyadic Green's functions in terms of the normal modes is obtained by substitution of a Green's function and a corresponding normal mode function into Green's second vector identity and by application of the homogeneous boundary conditions. The resultant magnetic Green's functions in M. K. S. units is:

$$\Gamma^{(2)}(r,s) = -j\omega \left\{ \sum_a \frac{\bar{F}_a(r)\, \bar{F}_a(s)}{\omega_a^{\,2} - \omega^2} - \frac{1}{\omega^2} \sum_k \bar{F}_k(r)\, \bar{F}_k(s) \right\}. \qquad (6)$$

The $\bar{F}_a(r)$'s are the magnetic mode vectors corresponding to the eigen-values ${}_a$ of the vector wave equations for $\bar{H}(r)$ in the interior region. These orthonormal mode vectors have zero divergence. Hence the additional vectors $\bar{F}_k(r)$ having zero curl are required for completeness, but can be omitted for the radiation field. The normal mode vectors are orthogonal and are normalized so that

$$\int_V \mu\, \bar{F}_a(r) \cdot \bar{F}_b(r)\, dV = \begin{cases} 1 & a = b \\ 0 & a \neq b \end{cases} \qquad (7)$$

7. Levine and Schwinger, ibid, pp 363-366.

This magnetic Green's function $\Gamma^{(2)}(r, s)$ is compared with the admittance dyadic $\mathbf{Y}(r, s)$ of Marcuvitz[8] and Schwinger in Appendix B.

8. Marcuvitz, N. and Schwinger, J. "On the Representation of the Electric and Magnetic Fields Produced by Currents and Discontinuities in Wave Guides", Jour. Appl. Phys., Vol. 22, June, 1951, pp 806-819.

The Effect of Wall Losses

The input admittance determined from the Green's function has a singularity at each resonant frequency. This means that the calculated input admittance will be infinite at the point where a real resonator will have a finite input admittance. This means that the theoretical calculations using the Green's functions fail to give a finite answer just at the region where experimental data can be observed. If a Green's function could be found for the boundary condition of lossy walls, the regions where theoretical and experimental results could be obtained would overlap and thus permit comparison. The wall losses in a cavity resonator at a given frequency are determined by the total field as follows:

$$P_{wall} = (1/2) \iint_s \left\{ \sum_a a_a \left(\bar{n} \times \bar{H}_a \right) \right\} \cdot \left\{ \sum_b a_b \left(\bar{n} \times \bar{H}_b \right) \right\} \sqrt{\frac{\omega\mu}{2\sigma}}\, dS. \qquad (8)$$

Examination of this equation shows that the contributions of the different modes add in quadrature, not linearly, so that an exact calculation of wall losses would be very cumbersome.

When the wall losses are relatively small, as is usually the case in microwave resonators, the boundary condition for lossy walls can be set up as a perturbation of boundary conditions similar to that of Feshbach[9].

9. Feshbach, H. "On the Perturbation of Boundary Conditions", Phys. Rev, Vol. 65, June 1 and 15, 1944, pp 307-318.

For a resonator with lossy walls, the boundary condition on H becomes:

$$\bar{n} \times \operatorname{curl} \bar{H} = \left[\bar{n} \times \bar{H}\right] \cdot \overleftrightarrow{Z} \tag{9a}$$

where

$$\overleftrightarrow{Z} = j\omega\epsilon \overleftrightarrow{\zeta} \sqrt{\frac{j\omega\mu}{\sigma}}\,, \tag{9b}$$

$$\overleftrightarrow{\zeta} = \begin{bmatrix} O & +\bar{a}_x\,\bar{a}_y \\ -\bar{a}_y\,\bar{a}_x & O \end{bmatrix}, \tag{9c}$$

and σ is the wall conductivity. The unit vectors on the boundary surface are $\bar{a}_x$ and $\bar{a}_y$ respectively. The differential equation is unchanged, but the boundary condition becomes:

$$\bar{n}_r \times \operatorname{curl}_s \overleftrightarrow{\Gamma}^{(2)}(r,s) = \left[\bar{n}_s \times \overleftrightarrow{\Gamma}^{(2)}(r,s)\right] \cdot \overleftrightarrow{Z} \tag{10}$$

Substituting H(r) and $\overleftrightarrow{\Gamma}^{(2)}(r,s) \cdot \bar{c}$ into Green's second vector identity equation and using the wave equations with the differential equation and boundary conditions (9) for the Green's functions give:

$$H(s) = \left\{ \begin{aligned} &\int_s \left[\bar{E}(r) \times \bar{n}_r\right] \cdot \overleftrightarrow{\Gamma}^{(2)}(r,s)\, dS(r) \\ &+ \frac{1}{j\omega\epsilon} \int_s \bar{H}(r) \cdot \overleftrightarrow{Z} \cdot \left[\bar{n}_r \times \overleftrightarrow{\Gamma}^{(2)}(r,s)\right] dS(r) \end{aligned} \right\}. \tag{11}$$

Equation (11) can also be considered as an integral equation for the magnetic Green's function for lossy walls. If $\bar{E} \times \bar{n}$ and $\bar{H} \times \bar{n}$ are known over the boundary then the problem would be the solution of this integral equation for $\overleftrightarrow{\Gamma}^{(2)}(r,s)$. In a resonator coupling problem the field usually can be assumed as known only at the iris, so there is insufficient information available to use this integral equation.

Since a Green's function does not have to obey the same boundary conditions as the corresponding field, a Green's function satisfying equation (4) may be used. However, this still does not resolve the problem, because the field must still be known over the whole boundary. It would simplify the problem if a modified Green's function could be found which has modified characteristic admittance terms representing the effect of wall losses. Such a modified Green's function would reduce equation (11) to the simpler form of equation (5).

When the wall losses are accounted for by the characteristic admittance terms in the Green's function, the field distribution in the resonator can be replaced by the corresponding field distribution for infinitely conducting walls. Such an approximation would simplify the problem, since $\bar{E} \times \bar{n}$ would need to be known only at the iris. This procedure would fail to give the local variations in the field inside the volume due to the wall losses. However, it would give the total effect of the wall losses upon the input admittance of the resonator as seen from a distance down the waveguide from the coupling iris.

If the wall losses are small, there are two methods using a perturbation of the boundary conditions that can be tried to obtain such a modified Green's function: (1) Repeated substitution of successive approximations to the Green's function into an integral equation, using the Green's function for infinitely conducting walls as the first approximation[9]; (2) Substitution of perturbed eigenvalues for lossy walls into the Green's functions for infinitely conducting walls.

The first of the above methods results in a sum of conditionally convergent series, which are different for $\omega < \omega_a$ and $\omega > \omega_a$. This method avoids any cumulative violation of equation (6) in the summation of the wall losses, but in so doing the admittance near each resonance fails to correspond to physical reality.

The second method shifts the singularities to the left of the $j\omega$ — axis in the complex frequency plane $(\alpha + j\omega)$ so that the input admittance is real and finite at each resonance, which corresponds to the range of experimentally observable phenomena. This approximation, however, violates equation (6).

An examination of these two methods of approximating the Green's functions for lossy walls shows that the second method in which the perturbed eigenvalues are used, is a reasonable approximation for microwave cavity resonators where the internal Q's are normally high.

Approximate Green's Functions Using Perturbed Eigenvalues

The approximate perturbed eigenvalue can be determined as follows by the method used by Feshbach[9]. Where $\overline{H}(r)$ is the magnetic field in the lossy case and $\overline{F}(r)$ is the non-lossy orthonormal mode magnetic field vector for the unperturbed eigenvalue ω_α, the wave equations are

$$\text{curl curl } \overline{H} = (k')^2\, \overline{H}, \tag{12a}$$

$$\text{curl curl } \overline{F}_\alpha = k_\alpha^2\, \overline{F}_\alpha, \tag{12b}$$

and the boundary conditions are

$$\overline{n} \times \text{curl } \overline{H} = [n \times H] \cdot \overleftrightarrow{\zeta}\, j\omega\epsilon Z_s, \tag{13a}$$

$$\overline{n} \times \text{curl } \overline{F}_\alpha = 0. \tag{13b}$$

Putting $\overline{H}$ and $\overline{F}_\alpha$ into Green's second vector identity and using the above wave equation and boundary conditions give

$$\left\{k^2 - (k')^2\right\} \int_V \overline{H} \cdot \overline{F}_\alpha \, dV = j\omega\epsilon \int_S [\overline{n} \times \overline{H}] \cdot \overleftrightarrow{\zeta} \cdot \overline{F}_\alpha \, Z_s \, dS. \tag{14}$$

By taking $\overline{F}_\alpha(r)$ as the first approximation to $\overline{H}(r)$, the perturbed eigenvalue or resonant frequency is

$$(\omega')^2 = \omega_\alpha^2 + j\omega F_{\alpha\alpha}, \tag{15}$$

$$F_{\alpha\beta} = (1 + j)\sqrt{\omega_\alpha \omega_\beta} \,/\, Q_{\alpha\beta}, \tag{16}$$

$$Q_{\alpha\beta}(\omega) = \frac{\sqrt{\left(\int_V \frac{\mu F_\alpha^2}{2} dV\right)\left(\int_V \frac{\mu F_\beta^2}{2} dV\right)}}{\frac{1}{2}\int_S \sqrt{\frac{\omega\mu}{2\sigma}} \left(\overline{n} \times \overline{F}_\alpha\right)\cdot\left(\overline{n} \times \overline{F}_\beta\right) dS}$$

when $\alpha = \beta$, $Q_{\alpha\alpha} = \omega U_H / P_L$

where U_H is the peak energy stored in the magnetic field and P_L is the power loss in the walls.

Using the expansion in normal modes for the magnetic Green's function for infinitely conducting walls from equation (6), and neglecting the $\overline{F}_k(r)$ terms of the local non-radiation fields at the sources, and replacing the eigenvalues ω by the perturbed eigenvalues ω_a' from equation (15) gives

$$\overleftrightarrow{\Gamma}^{(2)}(r,s) = -j\omega\epsilon \sum_a \frac{\overline{F}_a(r)\,\overline{F}_a(s)}{\left\{\omega_a^2 - \frac{\omega\omega_a}{Q_{aa}}\right\} - \omega^2 + j\frac{\omega\omega_a}{Q_{aa}}}. \qquad (17)$$

This approximation to the Green's function for lossy walls is limited in its use by the restrictions (1) that Q_{aa} must be large and (2) that it cannot be used for modes which are close together.

STATIONARY INTEGRAL EQUATION

Integral Equation

Applying Poyntings theorem to volume A of figure 1 gives:

$$-\int_S \overline{n}_A \cdot (\overline{E} x \overline{H})\ dS = -\int_V \overline{E}\cdot\overline{J}\ dv - \int_V \left(\overline{E}\cdot\frac{\partial \overline{D}}{\partial t} - \overline{H}\cdot\frac{\partial \overline{E}}{\partial t}\right) dv.$$

The wall losses are temporarily neglected, since their effect has been in the modified Green's functions developed in the previous section. By assuming the generating field is in the waveguide to the left of plane 2-2', the right hand integrals reduce to zero, giving

$$\int_S \overline{n}_A \cdot (\overline{H} x \overline{E})\ dS = \int_S \overline{H} \cdot (\overline{n}_A x \overline{E})\ dS = 0. \qquad (18)$$

Figure 1 — Iris Coupling between a Waveguide and a Cavity Resonator

The magnetic field in the iris contains the following components:

$$H_{iris} = H^{o}_{\substack{incident\\waveguide}} + H^{o}_{\substack{reflected\\waveguide}} + H'_{\substack{reflected\\waveguide}} + H_{\substack{absorbed\\resonator}} \qquad (19)$$

where the superscript ' o ' indicates principal mode and ' indicates higher modes.

Defining: $I_i \ V_i = \int_S \overline{H}_i \cdot (\overline{n}_i \times \overline{E}_i) \ dS,$ (20a)

proportional to the power into the region

and defining: I_i by $\overline{H}_i = I_i \ \overline{h}_i$, give: (20b)

$$V_i = \int_{iris} \overline{h}_i \cdot (\overline{n}_i \times \overline{E}_i) \, dS = \int_{iris} \overline{h}_i \cdot (\overline{n}_i \times \overline{E}) \, dS. \quad (21)$$

The replacement of $\overline{E}_i$ by $\overline{E}$ in equation (21) is permitted by the orthogorality properties of the normal mode vectors.

Letting: $I^o = I^O_{inc} + I^o_{refl}$, and substituting equations (19) and (21) into (18) give:

$$I^o V^o_B = -\left\{ - \int_{iris} \overline{H}'_{refl} \cdot (\overline{n}_B \times \overline{E}) \, dS + \int_{iris} \overline{H}_{abs} \cdot (\overline{n}_A \times \overline{E}) \, dS \right. . \quad (22)$$

Substituting the integral of equation (5) in place of $\overline{H}'_{refl}$ and $\overline{H}_{abs}$ in equation (22) gives:

$$I^o V^o_B = -\left\{ \begin{array}{l} - \int dS_r \int dS_s \left[\overline{n}_B \times \overline{E}(r) \right] \cdot \overleftrightarrow{\Gamma}_B^{(2)}(r,s) \cdot \left[\overline{n}_B \times \overline{E}(s) \right] \\ + \int dS_r \int dS_s \left[\overline{n}_A \times \overline{E}(r) \right] \cdot \overleftrightarrow{\Gamma}_A^{(2)}(r,s) \cdot \left[\overline{n}_A \times \overline{E}(s) \right] \end{array} \right\}. \quad (23)$$

A prime on a Green's function indicates that the dominant mode term is omitted.

The relative admittance which would be observed by measurement of the standing wave ratio and the position of the voltage minimum at a point to the left of plane 2-2' is:

$$Y/Y_o = I^o / Y_o \ V^o_B = I^o \ V^o_B / Y_o \ (V^o_B)^2 \quad (24)$$

The characteristic admittance Y_o of the waveguide is defined to be consistent with Y_i in $\overleftrightarrow{\Gamma}_B^{(2)}(r,s)$ as defined in Appendix A.

Combining equations (28) and (29) and using the following principal mode voltages

$$V_B^o = \int_{iris} \left[\bar{n}_B \times \bar{E}(r)\right] \cdot \bar{h}_{oB}\, dS(r) \qquad \text{(waveguide)} \qquad (25a)$$

$$V_A^o = \int_{iris} \left[\bar{n}_A \times \bar{E}(r)\right] \cdot \bar{F}_{oA}\, dS(r), \qquad \text{(resonator)} \qquad (25b)$$

give the following integral equation:

$$\frac{Y}{Y_o} = \frac{\omega\mu}{K_{1,O}} \left\{ \begin{array}{l} + \dfrac{\int dS_r \int dS_s \left[\bar{n}_B \times \bar{E}(r)\right] \cdot \overleftrightarrow{\Gamma}_B^{(2)}(r,s) \cdot \left[\bar{n}_B \times \bar{E}(s)\right]}{(V_B^o)^2} \\ + \left(\dfrac{V_A^o}{V_B^o}\right)^2 \left[\begin{array}{l} - \dfrac{\int dS_r \int dS_s \left[\bar{n}_A \times \bar{E}(r)\right] \cdot \overleftrightarrow{\Gamma}_A^{(2)'}(r,s) \cdot \left[\bar{n}_A \times \bar{E}(s)\right]}{(V_A^o)^2} \\ + \dfrac{j\omega}{\omega_o^2 - \omega^2 + \dfrac{\omega\omega_o}{Q_o}} \end{array} \right] \end{array} \right\}$$

$$= \frac{1}{Y_o} \left\{ \begin{array}{l} + Y_B' \\ + N^2 \left[\begin{array}{l} + Y_A' \\ + Y_{Ao} \end{array} \right] \end{array} \right\} \qquad (26)$$

The eigenfunctions appearing in the magnetic Green's functions in the above equation are defined in Appendices A and B.

Stationary Properties of the Admittance Components

Equation (26) is an integral equation involving two unknowns, the input admittance Y and the iris field $\bar{E}(r)$. If a good approximation for the iris field can be obtained, it can be substituted under the integral signs to give an approximation for the input admittance.

The principal mode admittance function Y_{AO} of the resonator, is independent of the choice of the trial field $\overline{E}(r)$. The susceptances in the resonator and waveguide $Y_A' = jB_A'$ and $Y_B' = jB_B'$ respectively, both have the same form which will be shown to be in a variational form. To simplify the notation, let

$$\overline{K}_m(r) = \overline{E}(r) \times \overline{n} \ . \qquad (27)$$

The stationary property Y_A' and Y_B' will be given briefly in the manner developed by J. S. Schwinger[10]. Two additional equations are required. First from equation (5) with definition (20b) comes:

$$H(s) = I\,\overline{h}(s) = \int_S \overline{K}_m(r)\cdot \overleftrightarrow{\Gamma}^{(2)}(r,s)\, dS(r). \qquad (28)$$

Second, from the Poynting vector and equation (21),

$$\frac{1}{Y} = \frac{I\,V}{I^2} = \frac{\int \overline{n}\cdot(\overline{E}\times\overline{H})\, dS}{I^2} = \frac{\int_S \overline{K}_m(r)\cdot\overline{h}(r)\, dS(r)}{I} \ . \qquad (29)$$

Then the result of multiplying equation (28) by $\overline{K}_m(s)$ and integrating over the surface S again and dividing twice by equation (21) gives another equation for Y which is identical with Y_A' and Y_B' in equation (26):

$$Y = \frac{I\,V}{V^2} = \frac{\int dS(r)\int dS(s)\ \overline{K}_m(r)\cdot \overleftrightarrow{\Gamma}^{(2)}(r,s)\cdot \overline{K}_m(s)}{\left[\int \overline{K}_m(r)\cdot \overline{h}(r)\, dS(r)\right]^2} \qquad (30)$$

By setting the current I equal to one, equation (29) becomes an integral equation for the determination of the aperture field $\overline{K}_m(r)$ and simplifies equation (30). This equation has been shown by Schwinger to be stationary with respect to first order variations of $\overline{K}_m(r)$. This stationary property of the variational integral equation (30) permits the use of any reasonable approximation for the iris field to obtain a good approximation since the error in Y is at most proportional to the square of the error in $\overline{K}_m(r)$

10. David Saxon, "Notes on Lectures of Julian Schwinger," Discontinuities in Waveguides. M.I.T. Rad. Lab. (unpublished) 1945. pp. 11-12, 44-46.

The remaining factor to be examined is N^2 from equation (26),

$$N^2 \times \left[\int \bar{K}_m \cdot \bar{h}_{o\beta} \, dS \right]^2 = \left[- \int \bar{K}_m \cdot \bar{F}_o A \, dS \right] \tag{31}$$

where N_o^2 is the exact N^2 and $\delta(N^2)$ is the error caused by the error δK in the assumed iris field

$$N^2 = N_o^2 + \delta(N^2). \tag{32a}$$

$$\bar{K}_m(r) = \bar{K}_o(r) + \delta \bar{K}(r) \tag{32b}$$

Substitution of equation (32a) and (32b) into (31) gives the error in N^2:

$$\frac{\delta(N^2)}{N_o^2} = 2 \left[\frac{\int \delta \bar{K}(r) \cdot \bar{F}(r) \, dS}{\int \bar{K}_o(r) \cdot \bar{F}(r) \, dS} - \frac{\int \delta \bar{K}(r) \cdot \bar{h}(r) \, dS}{\int \bar{K}_o(r) \cdot \bar{h}(r) \, dS} \right]. \tag{33}$$

The ratio N^2 is stationary with respect to first order variations of $\bar{K}_m(r)$ if:

$$\bar{F}(r) = A \, \bar{h}_t(r) \qquad \text{in the iris.} \tag{34}$$

Here A is an arbitrary constant, which cancels out, since it appears in both numerator and denominator. Thus $\delta(N^2) = 0$, if the tangential components of the principal magnetic normal mode functions have the same form in both resonator and waveguide.

The integral equation for the input admittance, equation (20), consists of elements which are stationary with respect to first order variations of the iris electric field $\bar{E}(r)$, when the principal magnetic normal mode functions have the same form in both waveguide and resonator.

Equivalent Circuit

Examination of equation (26) shows that this integral equation can be represented by the equivalent circuit of figure 2. The principal mode elements in the resonator are:

$$L_o = 1, \qquad C_o = \frac{1}{\omega_o^2} \qquad R_o = \frac{\omega_o}{Q_o}, \qquad Q_o = \frac{\omega L_o}{R_o}$$

$$Y_{AO} = \frac{1}{R + j\left(\omega L_o - \frac{1}{\omega C_o}\right)} = \frac{j\omega}{\left(\omega_o^2 - \omega^2 + j\frac{\omega\omega_o}{Q_o}\right)}.$$

Figure 2 - Equivalent Circuit

The input admittance of equation (31) has been obtained by setting up an integral equation at the iris plane 1-1' which gives the input admittance at a plane 2-2' an integral number of half-wavelengths down the waveguide where the higher waveguide modes are negligible.

CALCULATION OF THE COUPLED Q

Trial Iris Field

A reasonable approximation to the field in a rectangular iris is

$$E(x,y) = \bar{a}_y \cos \frac{\pi}{2c}(x - a/2) \Big/ \left[1 - \left(\frac{2}{d}\right)^2 \left(y - b/2\right)^2\right]^{1/2} \qquad (35)$$

The numerator corresponds to the cosine variation of the electric field in the principal mode. The denominator makes the field infinite at the top and bottom edges, as is the case in diffraction at a sharp edge. Insertion of equation (35) into the stationary integral equation (26) gives an approximate value of the admittance of the resonator.

Coupled Q_c

When the internal Q due to wall loss is high so that the resonator can be approximated by a lossless network, the stored energy can be calculated from the slope of the susceptance curve at a zero, which gives[11]

$$Q_c = \frac{\omega}{2Y_o} \frac{\partial B}{\partial \omega} . \qquad (36)$$

To obtain the coupled Q, equation (31) is used in a manner similar to that given by Montgomery[11]:

$$Q_c = \frac{\omega}{2Y_o} \frac{\partial B}{\partial \omega} = \frac{\pi \lambda_g L}{\lambda^2} \left[1 + \left(\frac{B_m}{Y_o}\right)^2\right] \qquad (37)$$

Equation (37) is not restricted by the conditions $\simeq \lambda_g/2$ and $(B/Y_o)^2 \gg$ used by Montgomery.

Extension of Bethe's Lumped Constants to "v-Factor" and "q-Factor" Curves.

For the small irises considered by Bethe[12], the same lumped con-

11. Montgomery, C.G. Principles of Microwave Circuits, 1948. pp 230-234, Equation (41) and (57).

stants could be used to determine Q_c and Δf_O. For large irises the Q and the change in resonant frequency curves do not maintain the same functional relationship, so it is more convenient to define separate functions. For comparison with experimental results and with other analyses, it is convenient to define a "q-factor" and a "v-factor" which are functions of iris dimensions. For small irises these factors reduce to the following equations where M is the magnetic polarizability of Bethe[12] in M. K. S. units.

$$q = (\mu / M)^2 \quad (38a) \qquad v = (M/\mu)^1 \quad (38b)$$

These factors are defined by the following equations:

$$\Delta\omega / \omega = - v / 4 N_r , \tag{39}$$

$$Q_c = N_g \cdot q \cdot N_r . \tag{40}$$

Where:

$$N_g = S_a / \omega \mu |H|^2 \quad \text{(waveguide)} \tag{41}$$

$$N_r = U_e / \mu |H|^2 \quad \text{(resonator)} \tag{42}$$

$$S_a = (1/2) \int (\bar{\ } \times \bar{E}_a) \cdot \bar{H}_a \, dS \tag{43}$$

For a $TE_{1,o}$ waveguide: $N_g = (1 \ \pi)\, ab\, \lambda_g$ (44)

For a $TEM_{O,C,3}$ resonator: with iris centered at

$$u = L/3 : \; N_r = (\pi/2)\, r_2^2 \ \ln (r_2 / r_1) \tag{45}$$

For the waveguide and resonator of figure 4 at $f = 2820$ mc/s. :

$$N_g = 123 \times 10^{6} \text{(meter)}, \; N_r = 89.7 \times 10^{6} \text{(meter)}.$$

This form of representing Q_c and $\Delta\omega/\omega$ is useful in using results from one resonator in obtaining approximate solutions for resonators having similar field distribution at the iris.

12. Bethe, H. A. "Lumped Constants for Small Irises", M. I. T. Radiation Laboratory, Report 194 (43-22), Mar 24, 1943, (PB 2839).

Rectangular Resonator

A rectangular resonator can be considered as a shorted waveguide, so the Green's functions are reduced from triple to double sums. The dyadic Green's function becomes:

$$\overleftrightarrow{\Gamma}^{(2)}(r,s) = \sum_{m} \sum_{n} Y^{o}{}_{m,n} \coth \gamma_{m,n} L \, \overline{H}_{m,n}(r) \, \overline{H}_{m,n}(s). \tag{44}$$

The normal mode functions $\overline{H}_{m,n}(r)$ are defined in Appendix B.

Figure 3 - Simplified Rectangular Resonator

Dividing the problem into two parts, the admittance on the waveguide side of the iris is approximately one-half of the admittance of the same iris in an infinite waveguide. The dimensions of the resonator in Figure 3 were chosen so that the susceptance curves of Marcuvitz[13] could be used.

Letting $\coth \gamma_{m,n} L$ equal $\cot K L$ and one respectively for the principal mode and higher modes in equation (44), reduces equation (26) to the following for this case:

13. Marcuvitz, N, "The Representation, Measurement and Calculation of Equivalent Circuits for Waveguide Discontinuitie with Application to Rectangular Slots." Microwave Research Institute, Polytechnic Institute of Brooklyn, New York (1949), Report R-193-49, PIB-137. Figures 5.3 and 5.4

$$\frac{Y}{Y_0} = \frac{1}{Y_0}\left[G_{1,0,1} + j\left(B_m - \cot \kappa_{1,0} L \right)\right] \quad (45)$$

By use of the curves of B_m from Marcuvitz and the condition that the imaginary part of the input admittance function of equation (45) in zero at resonance, the curves of shift in resonant frequency as a function of iris dimensions were calculated. The curves of Δf were converted to v-factor curves as plotted in figure 5 by use of equation (39).

Curves of Q_c were then calculated from equation (37) using the calculated from the curves of figure 5 and the corresponding curves of susceptance (B_m / Y_0) from Marcuvitz. The curves of Q_c for this rectangular resonator were then converted to q-factor by use of equation (40) and are plotted in figure 6 as curves "A".

Coaxial Resonator

For a rectangular waveguide coupled to a coaxial resonator as shown in figure 4, the integral equation is the same as equation (26). Here $\overleftrightarrow{\Gamma}_B^{(2)}(r, s)$ is the shorted waveguide Green's function of equation (44) with $\coth \gamma_{m,n} L = 1$, since the waveguide is considered as going to infinity on the left, and $\overleftrightarrow{\Gamma}_A^{(2)}(r,s)$ is the resonator Green's function of equation (6) where the $\overline{F}_\alpha(r)$'s are the coaxial resonator normal mode vectors of Appendix A.

For coupling between the $TE_{1,0}$ -mode rectangular waveguide and the $EM_{0,0,3}$- mode coaxial resonator of figure 4, the elements of equation (26) are:

$$Y'_A = \left\{ \underset{(TEM)}{\sum_{\ell=0}^{\infty}{}'\, Y_{A_\ell}\left(\frac{V_{A_\ell}}{V_{A_0}}\right)^2} + \underset{(TE)}{\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\sum_{\ell=1}^{\infty} Y''_{A_{n,m,\ell}}\left(\frac{V''_{A_{n,m,\ell}}}{V_{A_0}}\right)^2} + \underset{(TM)}{\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\sum_{\ell=1}^{\infty} Y'_{A_{n,m,\ell}}\left(\frac{V_{A_{n,m,\ell}}}{V_{A_0}}\right)^2} \right\} \quad (46a)$$

$$Y'_B = \underset{(TE)}{\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}{}'\, Y''_{B_{m,n}}\left(\frac{V''_{B_{m,n}}}{V_{B_0}}\right)^2} + \underset{(TM)}{\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}{}'\, Y_{B_{m,n}}\left(\frac{V'_{B_{m,n}}}{V_{B_0}}\right)} \quad (46b)$$

The higher modes voltages $V_{A_{n,m,\ell}}$ and $V_{B_{m,n}}$ are defined by equations (25b) and (25a) by replacing the eigenfunctions with the appropriate functions from Appendices A and B.

Figure 4 - Coaxial Resonator

Figure 5 - Q-Factor as a Function of Iris Dimension

Figure 6 - Coupled Q as a Function of Iris Dimensions

A more detailed development and sample calculation are given elsewhere.[14]

A sample calculation of Q_c (α, 0.1) made using the coaxial resonator eigenfunctions in equation (46a) is plotted as curve B in figure 6. For large capacitive irises this direct calculation comes closer to the experimental curve than the rectangular resonator approximation for Q_c. Examination of the curves in figure 6 shows that for inductive irises there is no need to make a detailed calculation, because the rectangular resonator approximation gives good agreement.

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14. Library of Congress, Publication Board, PB 109671: F. B. Wood, "Coupling Between Waveguides and Cavity Resonators for Large Power Output", Electronics Research Laboratory, University of California, Interim Technical Report, Series No. 1, Issue No. 65, May 1, 1953.

APPENDIX A

GREEN'S FUNCTIONS FOR COAXIAL RESONATORS

For a coaxial resonator of inner and outer radii r_1 and r_2 and of length L, the magnetic dyadic Green's function is:

$$\overleftrightarrow{\Gamma}^{(2)}(r,s) = \left\{ \underset{(TEM)}{\sum_{\ell=1}^{\infty} Y_\alpha \overline{F}_\alpha(r)\,\overline{F}_\alpha(s)} + \underset{(TM)}{\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\sum_{\ell=1}^{\infty} Y'_\alpha \overline{F'}_\alpha(r)\,\overline{F'}_\alpha(s)} + \underset{(TE)}{\sum_{n=0}\sum_{m=1}\sum_{\ell=1} Y''_\alpha \overline{F''_\alpha}(r)\,\overline{F''_\alpha}(s)} \right\}$$

The subscript α is an abbreviation for n, m, ℓ.

$$Y_\alpha = \frac{j\omega}{\left(\omega_\alpha^2 - \omega^2 + j\dfrac{\omega\omega_\alpha}{Q_\alpha}\right)} \qquad (A.2)$$

The normal mode vectors are normalized so that

$$\int_V \epsilon\, \overline{E}_\alpha \cdot \overline{E}_\beta \, dV = \int_V \mu\, \overline{F}_\alpha \cdot \overline{F}_\beta \, dV = \begin{cases} 1 & \alpha = \beta \\ 0 & \alpha \neq \beta \end{cases} \qquad (A.3)$$

The normal mode vectors for the $TEM_{0,0,\ell}$ - modes are:

$$\bar{F}_{\ell}(r) = (2/\mu L)^{1/2} \cos \frac{\ell\pi u}{L} \bar{h}_{\ell}(r),$$

$$\bar{h}_{\ell}(r) = \bar{a}_{\phi} (2 \ln r_2/r_1)^{-1/2} \frac{1}{r}, \qquad (A.4)$$

$$\omega_{\ell}(\mu\epsilon)^{1/2} = K_{\ell} = \frac{\ell\pi}{L}$$

The normal mode vectors for the TM- and TE-modes are:

$$\text{TM:} \quad \bar{F}'_{\alpha}(r) = (2/\mu L)^{1/2} \cos \frac{\ell\pi u}{L} (\bar{a}_r h'_r + \bar{a}_{\phi} h'_{\phi}), \qquad (A.5)$$

$$\text{TE:} \quad \bar{F}''_{\alpha}(r) = \left\{ \frac{\lambda''_{\alpha}}{\lambda''_g} (2/\mu L)^{1/2} \cos \frac{\ell\pi u}{L} (\bar{a}_r h''_r + \bar{a}_{\phi} h''_{\phi}) \right.$$

$$\left. + \frac{\lambda''_{\alpha}}{\lambda''_c} (2/\mu L)^{1/2} \sin \frac{\ell\pi u}{L} \bar{a}_u h''_u \right\} \qquad (A.6)$$

The components of the normal mode vectors $\bar{h}_r$, $\bar{h}_{\phi}$, and $\bar{h}_u$ are obtained from H_r, H_{ϕ}, and H_z of the Waveguide Handbook[15] by dividing them by the current I_i, where

$$V_i = - j I_i Z_i \tan \frac{\ell\pi u}{L} \qquad (A.7)$$

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15. Marcuvitz, N. <u>Waveguide Handbook</u>, 1951. pp. 72 - 80.

except that the subscripts (n, m, ℓ) are used here to be consistent with the IRE standard nomenclature, while subscripts (m, n) are used in the _Waveguide Handbook_. The resonant frequencies are determined as follows:

$$\omega_{\alpha}^{2} \mu \epsilon = \frac{\ell^{2} \pi^{2}}{L^{2}} + \frac{4 \pi^{2}}{\lambda^{2}_{c_{\alpha}}} \qquad (A.8)$$

APPENDIX B

GREEN'S FUNCTION FOR RECTANGULAR WAVEGUIDE

For a short-circuited waveguide extending to infinity on the left the admittance dyadic is for $r \neq s$

$$\overleftrightarrow{\Gamma}^{(2)}(r,s) = \sum_m \sum_n Y_{m,n}\, \overline{H}^{(-)}_{m,n}(r)\, \overline{H}^{(-)}_{m,n}(s). \tag{B.1}$$

This is twice the $\overleftrightarrow{Y}(r,s)$ for a waveguide going to infinity at both ends given by Marcuvitz and Schwinger[16]. In the above formula the $H(\mp)$ $\underline{H}(+)$ terms have been replaced by $H(-)\,H(-)$, since the source and field points in this analysis are both at the short circuit. The normal mode functions h_x, h_y, and h_z are obtained by dividing the magnetic fields H_x, H_y, and H_z of Marcuvitz[17] by the current I_i. The zero of the coordinate system is at a corner of the waveguide, not at the center.

$$\kappa^2_{m,n} = k^2 - k^2_{c_{m,n}} \tag{B.2}$$

$$\text{TM (or E)-modes:} \qquad Y'_{m,n} = \frac{j\omega\epsilon}{\gamma} = \begin{cases} \dfrac{j\omega\epsilon}{\alpha_{m,n}} & f<f_c \\[2mm] \dfrac{\omega\epsilon}{\kappa_{m,n}} & f>f_c \end{cases} \tag{B.3}$$

$$\overline{H}^{(\pm)}(r) = (\bar{a}_x h'_{x_{m,n}} + \bar{a}_y h'_{y_{m,n}})\, e^{\pm j\kappa'_{m,n} z} \tag{B.4}$$

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16. Marcuvitz, N. and Schwinger, J., Jour. Appl. Phys., Vol. 22, June 1951. p. 818.

17. Marcuvitz, N. Waveguide Handbook 1951. pp. 57-60.

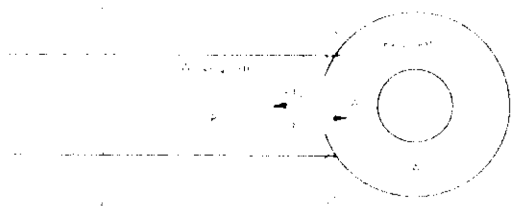
$$\text{TM (or H) - modes:} \quad Y''_{m,n} = \frac{\gamma}{j\omega\mu} = \begin{cases} \dfrac{\alpha_{m,n}}{j\omega\mu} & f < f_c \\ \dfrac{K_{m,n}}{\omega\mu} & f > f_c \end{cases} \qquad (B.5)$$

$$\overline{H}^{(\pm)}(r) = (\mp \bar{a}_x h''_{x_{m,n}} \mp \bar{a}_y h''_{y_{m,n}} + \bar{a}_z h''_{z_{m,n}})\, e^{\pm j K''_{m,n} z} \qquad (B.6)$$

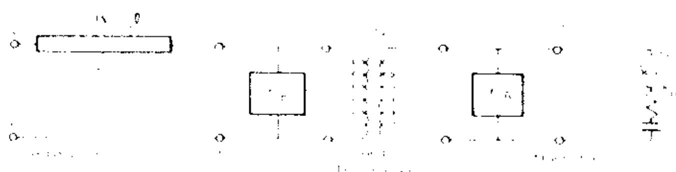
For a resonator consisting of a shorted section of rectangular waveguide of length L, $Y''_{m,n}$ must be multiplied by coth $\gamma_{m,n}L$.

FIGURE TITLES

1G 1



G 2



G 3

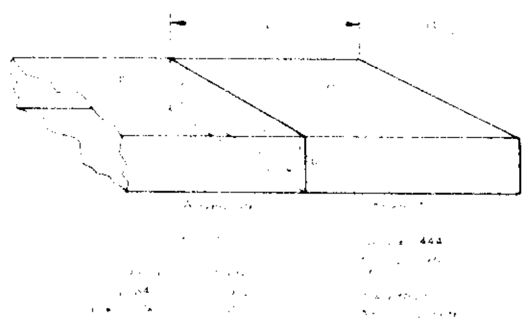


FIG 4

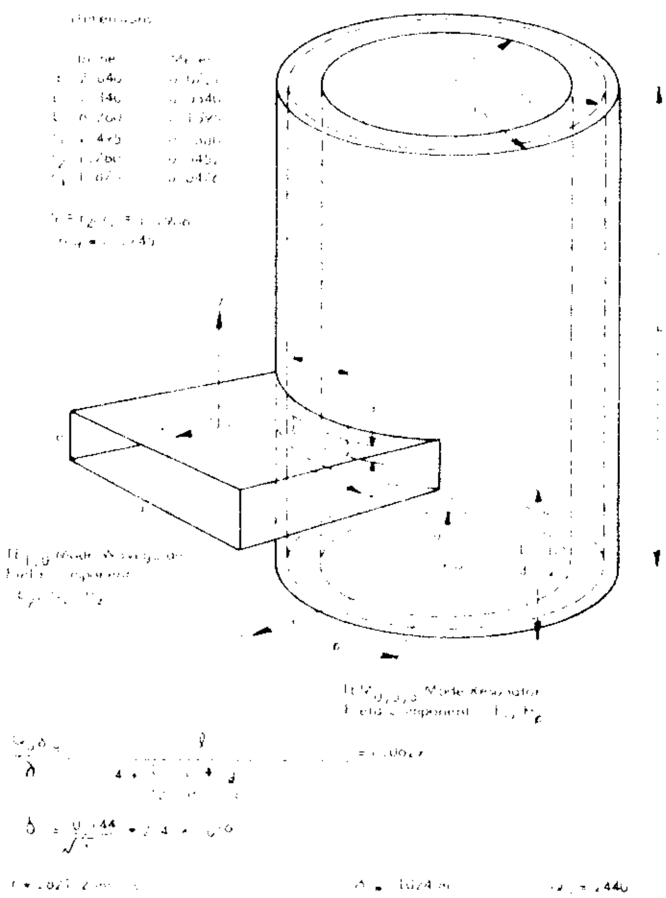


FIG 5

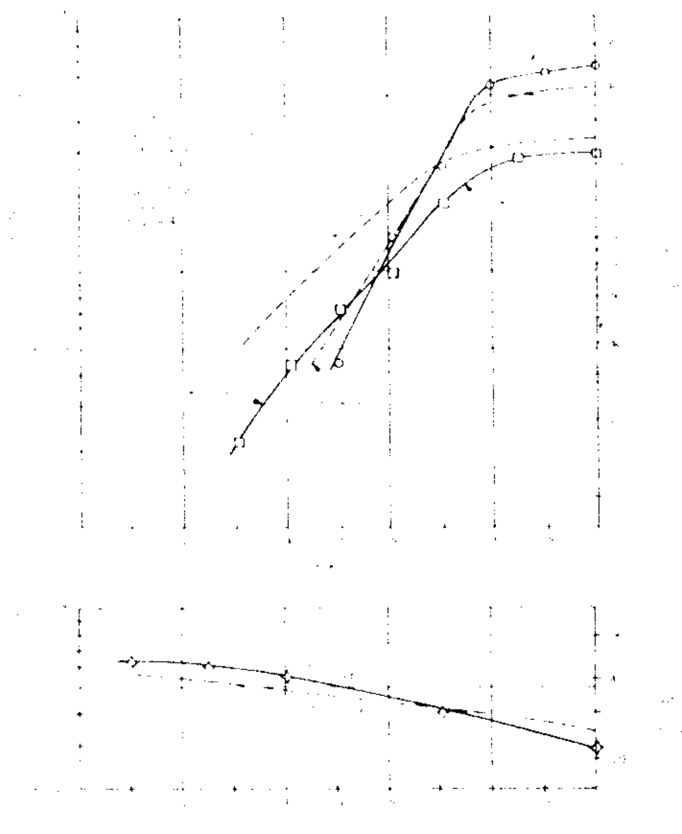


FIG 6